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Abstract

We prove that any positive solution of $\partial_t u - \Delta u + u^q = 0$ $(q > 1)$ in $\mathbb{R}^N \times (0, \infty)$ with initial trace $(F, 0)$, where F is a closed subset of $\mathbb{R}^N$ can be represented, up to two universal multiplicative constants, by a series involving the Bessel capacity $C_{2/q,q'}$. As a consequence we prove that there exists a unique positive solution of the equation with such an initial trace. We also characterize the blow-up set of $u(x,t)$ when $t \downarrow 0$, by using the "density" of F expressed in terms of the $C_{2/q,q'}$-Bessel capacity.

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Contents

1 Introduction

Let $T \in (0,\infty]$ and $Q_T = \mathbb{R}^N \times (0,T]$ $(N \geq 1)$. If $q > 1$ and $u \in C^2(Q_T)$ is nonnegative and verifies

$$\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_T, \tag{1.1}$$

it has been proven by Marcus and Véron [28] that there exists a unique outer-regular positive Borel measure ν in $\mathbb{R}^N$ such that

$$\lim_{t\to 0} u(.,t) = \nu, \tag{1.2}$$

in the sense of Borel measures; the set of such measures is denoted by $\mathfrak{B}^{reg}_+(\mathbb{R}^N)$. To each of its element ν is associated a unique couple $(\mathcal{S}_\nu, \mu_\nu)$ (we write $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$) where $\mathcal{S}_\nu$, *the singular part* of ν, is a closed subset of $\mathbb{R}^N$ and μ_ν, *the regular part* is a nonnegative Radon measure on $\mathcal{R}_\nu = \mathbb{R}^N \setminus \mathcal{S}_\nu$. In this setting, relation (1.2) has the following meaning :

$$\begin{array}{lll} (i) & \lim_{t\to 0} \int_{\mathcal{R}_\nu} u(.,t)\zeta dx = \displaystyle\int_{\mathcal{R}_\nu} \zeta d\mu_\nu, & \forall \zeta \in C_0(\mathcal{R}_\nu), \\ (ii) & \lim_{t\to 0} \displaystyle\int_{\mathcal{O}} u(.,t)dx = \infty, & \forall \mathcal{O} \subset \mathbb{R}^N \text{ open, } \mathcal{O} \cap \mathcal{S}_\nu \neq \emptyset. \end{array} \tag{1.3}$$

The measure ν is by definition *the initial trace* of u and denoted by $Tr_{\mathbb{R}^N}(u)$. It is wellknown that equation (1.1) admits a critical exponent

$$1 < q < q_c = 1 + \frac{N}{2}.$$

This is due to the fact, proven by Brezis and Friedman [7], that if $q \geq q_c$, isolated singularities of solutions of (1.1) in $\mathbb{R}^N \setminus \{0\}$ are removable. Conversely, if $1 < q < q_c$, it is proven by the same authors that for any $k > 0$, equation (1.1) admits a unique solution $u_{k\delta_0}$ with initial data $k\delta_0$. This existence and uniqueness results extends in a simple way if the initial data $k\delta_0$ is replaced by any Radon measure μ in $\mathbb{R}^N$ (see [6]). Furthermore, if $k \to \infty$, $u_{k\delta_0}$ increases and converges to a positive, radial and self-similar solution u_∞ of (1.1). Writing it under the form $u_\infty(x,t) = t^{-\frac{1}{q-1}} f(|x|/\sqrt{t})$, f is a positive solution of

$$\left\{\begin{array}{l} \Delta f + \frac{1}{2} y.Df + \frac{1}{q-1} f - f^q = 0 \quad \text{in } \mathbb{R}^N \\ \lim_{|y|\to\infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{array}\right. \tag{1.4}$$

The existence, uniqueness and the expression of the asymptotics of f has been studied thoroughly by Brezis, Peletier and Terman in [8]. Later on, Marcus and Véron proved in [28] that in the same range of exponents, for any $\nu \in \mathfrak{B}^{reg}_+(\mathbb{R}^N)$, the Cauchy problem

$$\left\{\begin{array}{ll} \partial_t u - \Delta u + u^q = 0 & \text{in } Q_\infty, \\ Tr_{\mathbb{R}^N}(u) = \nu, & \end{array}\right. \tag{1.5}$$

admits a unique positive solution. This result means that the initial trace establishes a one to one correspondence between the set of positive solutions of (1.1) and $\mathfrak{B}^{reg}_+(\mathbb{R}^N)$. A key step for proving the uniqueness is the following inequalities

$$t^{-\frac{1}{q-1}} f(|x-a|/\sqrt{t}) \leq u(x,t) \leq ((q-1)t)^{-\frac{1}{q-1}} \qquad \forall (x,t) \in Q_\infty, \tag{1.6}$$

valid for any $a \in \mathcal{S}_\nu$. As a consequence of Brezis and Friedman's result, if $q \geq q_c$, i.e. in the *supercritical range*, Problem (1.5) may admit no solution at all. If $\nu \in \mathfrak{B}^{reg}_+(\mathbb{R}^N)$, $\nu \approx (\mathcal{S}_\nu, \mu_\nu)$, the necessary and sufficient conditions for the existence of a maximal solution $u = \overline{u}_\nu$ to Problem (1.5) are obtained in [28] and expressed in terms of the the Bessel capacity $C_{2/q,q'}$, (with $q' = q/(q-1)$). Furthermore, uniqueness does not hold in general as it was pointed out by Le Gall [23]. In the particular case where $\mathcal{S}_\nu = \emptyset$ and ν is simply the Radon measure μ_ν, the necessary and sufficient condition for solvability is that μ_ν does not charge Borel subsets with $C_{2/q,q'}$-capacity zero. This result was already proven by Baras and Pierre [5] in the particular case of bounded measures and extended by Marcus and Véron [28] to the general case. We denote by $\mathfrak{M}^q_+(\mathbb{R}^N)$ the positive cone of the space $\mathfrak{M}^q(\mathbb{R}^N)$ of Radon measures which do not charge Borel subsets with zero $C_{2/q,q'}$-capacity. Notice that $W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ is a subset of $\mathfrak{M}^q_+(\mathbb{R}^N)$ where $\mathfrak{M}^b_+(\mathbb{R}^N)$ is the cone of positive bounded Radon mesures in $\mathbb{R}^N$. For such measures, uniqueness always holds and we denote $\overline{u}_{\mu_\nu} = u_{\mu_\nu}$.

In view of the already known results concerning the parabolic equation, it is useful to recall the main advanced results previously obtained for the stationary equation

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \tag{1.7}$$

in a smooth bounded domain Ω of $\mathbb{R}^N$. This equation has been intensively studied since 1993, both by probabilists (Le Gall, Dynkin, Kuznetsov) and by analysts (Marcus, Véron). The existence of a *boundary trace* for positive solutions, in the class of outer-regular positive Borel measures on $\partial\Omega$, is proven by Le Gall [22], [23] in the case $q = N = 2$, by probabilistic methods, and by Marcus and Véron in [26], [27] in the general case $q > 1$, $N > 1$. The existence of a critical exponent $q_e = (N+1)/(N-1)$ is due to Gmira and Véron [14] who shew that, if $q \geq q_e$ boundary isolated singularities of solutions of (1.7) are removable, which is not the case if $1 < q < q_e$. In this *subcritical case* Le Gall and Marcus and Véron proved that the boundary trace establishes a one to one correspondence between positive solutions of (1.7) in Ω and outer regular positive Borel measures on $\partial\Omega$. This fundamental result does not hold in the *supercritical case* $q \geq q_e$. In [12] Dynkin and Kuznetsov introduced the notion of σ-moderate solution which means that u is a positive solution of (1.7) such that there exists an increasing sequence of positive Radon measures on $\partial\Omega$ $\{\mu_n\}$ belonging to $W^{-2/q,q'}(\partial\Omega)$ such that the corresponding solutions $v = v_{\mu_n}$ of

$$\left\{\begin{aligned} -\Delta v + v^q &= 0 \quad \text{in } \Omega \\ v &= \mu_n \quad \text{in } \partial\Omega \end{aligned}\right. \tag{1.8}$$

converges to u locally uniformly in Ω. This class of solutions plays a fundamental role since Dynkin and Kuznetsov proved that a σ-moderate solution of (1.7) is uniquely determined by its *fine trace*, a new notion of trace introduced in order to avoid the non-uniqueness phenomena. Later on, it is proved by Mselati (if $q = 2$) [36], then by Dynkin (if $q_e \leq q \leq 2$) [10] and finally by Marcus with no restriction on q [25], that all the positive solutions of (1.7) are σ-moderate. One of the key-stones element in their proof (partially probabilistic) is the fact that the maximal solution $\overline{u}_K$ of (1.7) with a boundary trace vanishing outside a compact subset $K \subset \partial\Omega$ is indeed σ-moderate. This deep result was obtained by a combination of probabilistic and analytic methods by Mselati [36] in the case $q = 2$ and by purely analytic tools by Marcus and Véron [31], [32] in the case $q \geq q_e$. Defining $\underline{u}_K$ as the largest σ-moderate solution of

(1.7) with a boundary trace concentrated on K, the crucial step in Marcus-Véron's proof (non probabilistic) is the bilateral estimate satisfied by $\overline{u}_K$ and $\underline{u}_K$

$$C^{-1}\rho(x)W_K(x) \leq \underline{u}_K(x) \leq \overline{u}_K(x) \leq C\rho(x)W_K(x). \tag{1.9}$$

In this expression $C = C(\Omega, q)$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and $W_F(x)$ is the *elliptic capacitary potential* of K defined by

$$W_K(x) = \sum_{-\infty}^{\infty} 2^{-\frac{m(q+1)}{q-1}} C_{2/q,q'}(2^m K_m(x)), \tag{1.10}$$

where $K_m(x) = K \cap \{z : 2^{-m-1} \leq |z - x| \leq 2^{-m}\}$, the Bessel capacity being relative to $\mathbb{R}^{N-1}$. Note that, using a technique introduced in [27], inequality $\overline{u}_K \leq C^2 \underline{u}_K$ implies $\underline{u}_K = \overline{u}_K$.

The aim of this article is to initiate the fine study of the complete initial trace problem for positive solutions of (1.1) in the supercritical case $q \geq q_c$ and to give in particular the parabolic counterparts of the results of [36], [31] and [32]. Extending Dynkin's ideas to the parabolic case, we introduce the following notion

Definition 1.1 *A positive solution u of* (1.1) *is called σ-moderate if their exists an increasing sequence $\{\mu_n\} \subset W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ such that the corresponding solution $u := u_{\mu_n}$ of*

$$\left\{\begin{aligned} \partial_t u - \Delta u + u^q = 0 \quad & \text{in } Q_\infty \\ u(x,0) = \mu_n \quad & \text{in } \mathbb{R}^N, \end{aligned}\right. \tag{1.11}$$

converges to u locally uniformly in Q_∞.

If F is a closed subset of $\mathbb{R}^N$, we denote by $\overline{u}_F$ the maximal solution of (1.1) with an initial trace vanishing on F^c, and by $\underline{u}_F$ the maximal σ-moderate solution of (1.1) with an initial trace vanishing on F^c. Thus $\underline{u}_F$ is defined by

$$\underline{u}_F = \sup\{u_\mu : \mu \in W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), \mu(F^c) = 0\}, \tag{1.12}$$

(and clearly $W^{-2/q,q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ can be replaced by $\mathfrak{M}_+^q(\mathbb{R}^N)$). One of the main goal of this article is to prove that $\overline{u}_F$ is σ-moderate and more precisely,

Theorem 1.2 *For any $q > 1$ and any closed subset F of $\mathbb{R}^N$, $\overline{u}_F = \underline{u}_F$.*

We define below a set function which will play a fundamental role in the sequel.

Definition 1.3 *Let F be a closed subset of $\mathbb{R}^N$. The Bessel parabolic capacitary potential W_F of F is defined by*

$$W_F(x,t) = \frac{1}{t^{\frac{N}{2}}} \sum_{n=0}^{\infty} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \qquad \forall (x,t) \in Q_\infty, \tag{1.13}$$

where $C_{2/q,q'}$ is the N-dimensional Bessel capacity, $d_n = \sqrt{nt}$ and $F_n = \{y \in F : d_n \leq |x - y| \leq d_{n+1}\}$.

In our study, it is useful to introduce a variant of W_F with the help of the Besov capacity: if $\Omega \subset \mathbb{R}^N$ is a bounded domain, we set

$$\|\phi\|_{B_{2/q,q'}} = \left(\iint_{\Omega\times\Omega} \frac{|\phi(x)-\phi(y)|^{q'}}{|x-y|^{N+\frac{2}{q-1}}} dxdy\right)^{1/q'}, \tag{1.14}$$

if $1 < 2/q < 1$, and $\|\phi\|_{B_{1,2}} = \|\nabla\phi\|_{L^2}$ if $2/q = 1$ (i.e. $N = 2$ and $q = 2$). The Besov capacity of a compact set $K \subset \Omega$ relative to Ω is expressed by

$$R^{\Omega}_{2/q,q'} = \inf\left\{\|\phi\|^{q'}_{B_{2/q,q'}} : \phi \in C_0^\infty(\Omega), 0 \leq \phi \leq 1, \eta = 1 \text{ on K}\right\}. \tag{1.15}$$

The Besov-parabolic capacitary potential $\tilde{W}_F$ of F is defined by

$$\tilde{W}_F(x,t) = t^{-\frac{N}{2}} \sum_{n=0}^{\infty} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} R^{\Gamma_n}_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \qquad \forall (x,t) \in Q_\infty, \tag{1.16}$$

where $\Gamma_n = B_{d_{n+1}} \setminus \overline{B_{d_n}}$. The Besov-parabolic capacitary potential is equivariant with respect to the same scaling transformation which let (1.1) invariant in the sense that, for any $\ell > 0$,

$$\ell^{\frac{1}{q-1}} \tilde{W}_F(\sqrt{\ell}x, \ell t) = \tilde{W}_{F/\sqrt{\ell}}(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.17}$$

and we prove that there exists $c = c(N,q) > 0$ such that

$$c^{-1}\tilde{W}_F(x,t) \leq W_F(x,t) \leq c\tilde{W}_F(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.18}$$

One of the tool for proving Theorem 1.2 is the following bilateral estimate which is only meaningful in the supercritical case, otherwhile it reduces to (1.6);

Theorem 1.4 *For any $q \geq q_c$ there exist two positive constants $C_1 \geq C_2 > 0$, depending only on N and q such that for any closed subset F of $\mathbb{R}^N$, there holds*

$$C_2 W_F(x,t) \leq \underline{u}_F(x,t) \leq \overline{u}_F(x,t) \leq C_1 W_F(x,t) \qquad \forall (x,t) \in Q_\infty. \tag{1.19}$$

Actually our result is more general since the upper estimate in (1.19) is valid for *any* positive solution of

$$\partial_t u - \Delta u + u^q \leq 0 \qquad \text{in } Q_T \tag{1.20}$$

satisfying

$$\lim_{t\to 0} u(x,t) = 0 \qquad \text{locally uniformly in } F^c. \tag{1.21}$$

Extension to positive solutions of

$$\partial_t u - \Delta u + f(u) = 0 \qquad \text{in } Q_T \tag{1.22}$$

where f is continuous from $\mathbb{R}^+$ to $\mathbb{R}^+$ and satisfies

$$c_2 r^q \leq f(r) \leq c_1 r^q \qquad \forall r \geq 0 \tag{1.23}$$

for some $0 < c_2 \leq c_1$ is straightforward.

This *quasi representation*, up to uniformly upper and lower bounded functions, is also interesting in the sense that it indicates precisely *how to characterize the blow-up points* of $\overline{u}_F = \underline{u}_F := u_F$. Introducing an integral expression comparable to W_F, we show in particular the following results

$$\lim_{\tau\to 0} \tau^{\frac{2}{q-1}-N} C_{2/q,q'}\left(F\cap B_\tau(x)\right) = \gamma \in [0,\infty) \Longrightarrow \lim_{t\to 0} t^{\frac{1}{q-1}} u_F(x,t) = C\gamma \tag{1.24}$$

for some $C_\gamma = C(N,q,\gamma) > 0$, and

$$\limsup_{\tau\to 0} \tau^{\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{\tau}\cap B_1(x)\right) < \infty \Longrightarrow \limsup_{t\to 0} u_F(x,t) < \infty. \tag{1.25}$$

Our paper is organized as follows. In Section1 we recall some properties of the Besov spaces with fractional derivatives $B^{s,p}$ and their links with heat equation. In Section 2 we obtain estimates from above on $\overline{u}_F$. In Section 3 we give estimates from below on $\underline{u}_F$. In Section 4 we prove the main theorems and expose various consequences. In Appendix we derive a series of sharp integral inequalities.

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2 Estimates from above

Some notations. Let Ω be a domain in $\mathbb{R}^N$ with a compact C^2 boundary and $T > 0$. Set $B_r(a)$ the open ball of radius $r > 0$ and center a (and $B_r(0) := B_r$) and

$$Q_T^\Omega := \Omega\times(0,T), \quad \partial_\ell Q_T^\Omega = \partial\Omega\times(0,T), \quad Q_T := Q_T^{\mathbb{R}^N}, \quad Q_\infty := Q_\infty^{\mathbb{R}^N}.$$

Let $\mathbb{H}^\Omega[.]$ (resp. $\mathbb{H}[.]$) denote the heat potential in Ω with zero lateral boundary data (resp. the heat potential in $\mathbb{R}^N$) with corresponding kernel

$$(x,y,t)\mapsto H^\Omega(x,y,t) \quad (\text{resp.}(x,y,t)\mapsto H(x,y,t) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}}).$$

We denote by $q_c := 1 + \frac{N}{2}$, the Brezis-Friedman critical exponent.

Theorem 2.1 *Let $q \geq q_c$. Then there exists a positive constant $C_1 = C_1(N,q)$ such that for any closed subset F of $\mathbb{R}^N$ and any $u \in C^2(Q_\infty)\cap C(\overline{Q_\infty}\setminus F)$ satisfying*

$$\begin{aligned} &\partial_t u - \Delta u + u^q = 0 \quad \text{in } Q_\infty \\ &\lim_{t\to 0} u(x,t) = 0 \quad \text{locally uniformly in } F^c, \end{aligned} \tag{2.1}$$

there holds

$$u(x,t) \leq C_1 W_F(x,t) \qquad \forall (x,t)\in Q_\infty, \tag{2.2}$$

where W_F is the $(2/q,q')$-parabolic capacitary potential of F defined by (1.13).

First we consider the case where $F = K$ is compact and

$$K \subset B_r \subset \overline{B}_r, \tag{2.3}$$

and then we extend to the general case by a covering argument.

2.1 Capacities and Besov spaces

2.1.1 L^p regularity

Throughout this paper C will denote a generic positive constant, depending only on N, q and sometimes T, the value of which may vary from one occurrence to another. We also use sometimes the notation $A \approx B$ for meaning that there exists a constant $C > 0$ independent of the data such that $C^{-1}A \leq B \leq CA$.

We recall some classical results dealing with L^p capacities as they are developed in [5]: if $1 < p < \infty$ we denote

$$W_p^{2,1}(\mathbb{R}^{N+1}) := \{\phi \in L^p(\mathbb{R}^{N+1}) : \partial_t\phi, \nabla\phi, D^2\phi \in L^p(\mathbb{R}^{N+1})\}, \tag{2.4}$$

with the associated norm

$$\|\phi\|_{W_p^{2,1}} = \|\phi\|_{L^p} + \|\nabla\phi\|_{L^p} + \|\partial_t\phi\|_{L^p} + \left\|D^2\phi\right\|_{L^p}. \tag{2.5}$$

We define a corresponding capacity on compact sets, that we extend it classicaly on capacitable sets.

$$C_{2,1,p}(E) = \inf\{\|\phi\|_{W_p^{2,1}} : \phi \in C_0^\infty(\mathbb{R}^{N+1}) : \phi \geq 1 \text{ in a neighborhood of } E\}, \tag{2.6}$$

We extend the heat kernel H in $\mathbb{R}^{N+1} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R}\}$ by assigning the value 0 for $t < 0$. Then, for any $\eta \in C_0(\mathbb{R}^N)$,

$$\mathbb{H}[\eta](x,t) = \begin{cases} 0 & \text{if } t < 0 \\ H * (\eta \otimes \delta_0)(x,t) & \text{if } t > 0, \end{cases} \tag{2.7}$$

where δ_0 has to be understood as the Dirac measure on $\mathbb{R}$ at $t = 0$. For any subset $E \in \mathbb{R}^{N+1}$

$$C_{H,p}(E) = \inf\{\|f\|_{L^p} : f \in L^p(\mathbb{R}^{N+1}), H * f \geq 1 \text{ on } E\}. \tag{2.8}$$

The following result is proved in [5, Prop 2.1].

Proposition 2.2 *For any $T > 0$, there exists $c = c(T,p,N)$ such that*

$$c^{-1}C_{H,p}(E) \leq C_{2,1,p}(E) \leq cC_{H,p}(E) \qquad \forall E \subset \mathbb{R}^N \times]-T,T[,\ E \text{ Borel}. \tag{2.9}$$

We recall the Gagliardo Nirenberg inequality valid for any $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\|\nabla\phi\|_{L^{2p}}^{2p} \leq c_{d,p} \|\phi\|_{L^\infty}^p \left\|D^2\phi\right\|_{L^p}^p. \tag{2.10}$$

Furthermore, the trace at $t = 0$ of functions in $W^{2,1}_p$ belongs to the Besov space $B^{2-\frac{2}{p},p}(\mathbb{R}^N)$. However, in our range of exponents $B^{2-\frac{2}{p},p}(\mathbb{R}^N) = W^{2-\frac{2}{p},p}(\mathbb{R}^N)$. The reason for this is that $2 - \frac{2}{p}$ is not an integer except if $p = 2$, in which case equality holds also. If we set

$$c_{2-\frac{2}{p},p}(K) = \inf\{\|\phi\|_{W^{2-\frac{2}{p},p}} : \phi \in C_0^\infty(\mathbb{R}^N), \phi \geq 1 \text{ in a neighborhood of } K\}. \tag{2.11}$$

then [5, Prop 2.3].

Proposition 2.3 *There exist $c = c(N,p) > 0$ such that*

$$c^{-1}c_{2-\frac{2}{p},p}(E) \leq C_{2,1,p}(E \times \{0\}) \leq cc_{2-\frac{2}{p},p}(E) \qquad \forall E \subset \mathbb{R}^N,\ E \text{ Borel.} \tag{2.12}$$

The $c_{2-\frac{2}{p},p}$-capacity is equivalent to the Bessel capacity $C_{2-\frac{2}{p},p}$ defined by

$$C_{2-\frac{2}{p},p}(E) = \inf\{\|f\|_{L^p} : f \in L^p(\mathbb{R}^N), G_{2-\frac{2}{p}} * f \geq 1 \text{ on } E\} \tag{2.13}$$

where $G_{2-\frac{2}{p}} = \mathcal{F}[(1+|\xi|^2)^{\frac{1}{p}-1}]$ denotes the Bessel kernel associated to the operator $(-\Delta+I)^{1-\frac{1}{p}}$.

2.1.2 The Aronszajn-Slobodeckij integral

If Ω is a domain in $\mathbb{R}^N$ and $0 < s < 1$, we denote by $\|.\|_{\dot{B}^{s,p}(\Omega)}$ the Aronszajn-Slobodeckij norm defined on $C_0^\infty(\Omega)$ by

$$\|\eta\|_{\dot{B}^{s,p}} = \left(\iint_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p} \qquad \forall \eta \in C_0^\infty(\Omega). \tag{2.14}$$

In the case $1 < s < 2$, all the results which are presented still holds by replacing the function by its gradient. We also consider the case $s = 1$, but in our range of exponents the corresponding exponent for p is 2, in which case the space under consideration is just $H^1_0(\Omega)$. Since the imbedding of $W^{1,p}(\Omega)$ is compact, it follows the imbedding of $B^{s,p}(\Omega)$ into $L^p(\Omega)$ is compact too. Therefore the following Poincaré type inequality holds [39, p. 134]. Actually, the proof, obtained by contradiction, is given with $W^{1,p}(\Omega)$ instead of $B^{s,p}(\Omega)$, but it depends only on the compactness of the imbedding.

Proposition 2.4 *Let Ω be a bounded domain and, $p \in (1,\infty)$ and $0 < s \leq 1$ such that $sp \leq N$. Then there exists $\lambda = \lambda(\Omega, N, p) > 0$ such that*

$$\iint_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy \geq \lambda \int_\Omega |\eta(x)|^p dx \quad \forall \eta \in C_0^\infty(\Omega). \tag{2.15}$$

Remark. If $sp > N$, the same proof re holds for all $\eta \in C_0^\infty(\Omega)$ (see the proof of [9, Th 8.2])

$$\left(\iint_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p} \geq C\frac{|\eta(z)-\eta(z')|}{|z-z'|^\alpha} \quad \forall (z,z') \in \Omega\times\Omega,\ z \neq z', \tag{2.16}$$

with $\alpha = s - N/p$ and $C = C(s,N,p)$. This estimate implies

$$\left(\iint_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p} \geq Cd^{-\alpha}\left\|u\right\|_{L^\infty}, \tag{2.17}$$

where d is the width of Ω, i.e. the smallest of $\delta > 0$ such that there exists an isometry $\mathcal{R}$ such that $\mathcal{R}(\Omega) \subset D_\delta := \{x = (x_1, x') : 0 < x_1 < \delta\}$.

The related unpublished result due to L. Tartar [40] will be useful in the sequel. We reproduce its proof for the sake of completeness.

Proposition 2.5 *Assume $b > a$ and $\Omega \subset \Gamma_{a,b} := \{x = (x_1, x') : a < x_1 < b\}$ is a domain. If $sp \leq N$ there exists $C = C(s,p,N,b/a) > 0$ such that that*

$$\iint_{\Omega\times\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy \geq \lambda\,(b-a)^{sp}\int_\Omega |\eta(x)|^p dx \quad \forall \eta \in C_0^\infty(\Omega). \tag{2.18}$$

Proof. Using the notation of [24], $W^{s,p}(\mathbb{R}^N)$ is the interpolation space $[W^{1,p}(\mathbb{R}^N), L^p(\mathbb{R}^N)]_{s,p}$ and subset of $L^p(\mathbb{R}^{N-1}; [W^{1,p}(\mathbb{R}), L^p\mathbb{R}]_{s,p}) = L^p(\mathbb{R}^{N-1}; W^{s,p}(\mathbb{R}))$, with continuous imbedding. Thus there exist $C > 0$ such that

$$\begin{aligned} \|\eta\|_{L^p}^p + \int_{\mathbb{R}^{N-1}}\iint_{\mathbb{R}\times\mathbb{R}} \frac{|\eta(x_1,x')-\eta(y_1,x')|^p}{|x_1-y_1|^{1+sp}}dx_1dy_1dx' \\ \leq C\left(\|\eta\|_{L^p}^p + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy\right) \end{aligned} \tag{2.19}$$

for all $\eta \in C_0^\infty(\mathbb{R}^N)$. This inequality is valid if η is replaced by η_τ where $\eta_\tau(x) = \eta(\tau x)$ and $\tau > 0$. This gives

$$\begin{aligned} \|\eta\|_{L^p}^p + \tau^{sp-N}\int_{\mathbb{R}^{N-1}}\iint_{\mathbb{R}\times\mathbb{R}} \frac{|\eta(x_1,x')-\eta(y_1,x')|^p}{|x_1-y_1|^{1+sp}}dx_1dy_1dx' \\ \leq C\left(\|\eta\|_{L^p}^p + \tau^{sp-N}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy\right). \end{aligned}$$

Letting $\tau \to 0$, we obtained

$$\int_{\mathbb{R}^{N-1}}\iint_{\mathbb{R}\times\mathbb{R}} \frac{|\eta(x_1)-\eta(y_1)|^p}{|x_1-y_1|^{1+sp}}dx_1dy_1dx' \leq C\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+sp}}dxdy \qquad \forall \eta \in C_0^\infty(\mathbb{R}^N). \tag{2.20}$$

Using Proposition 2.4 with $N = 1$ we get

$$\int_0^1\int_0^1 \frac{|\eta(x_1,x')-\eta(y_1,x')|^p}{|x_1-y_1|^{1+sp}}|dx_1dy_1 \geq \lambda\int_0^1 |\eta(x_1,x')|^p dx_1 \quad \forall \eta \in C_0^\infty\left((0,1)\times\mathbb{R}^{N-1}\right)$$

for all $x' \in \mathbb{R}^{N-1}$. Using a standard change of scale, it transforms into

$$\int_a^b\int_a^b \frac{|\eta(x_1,x')-\eta(y_1,x')|^p}{|x_1-y_1|^{1+sp}}|dx_1dy_1 \geq \lambda(b-a)^{sp}\int_a^b |\eta(x_1,x')|^p dx_1 \quad \forall \eta \in C_0^\infty\left((a,b)\times\mathbb{R}^{N-1}\right)$$

Integrating over $\mathbb{R}^{N-1}$ and using (2.20), we derive (2.18). $\square$

Definition 2.6 *Assume $s \in (0,1)$ and $sp < 1$ or $s = 1$ and $p = 2$. If Ω is any domain in $\mathbb{R}^N$, the Besov space $B_0^{s,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm*

$$\|\eta\|_{B^{s,p}} = \|\eta\|_{\dot{B}^{s,p}} + \|\eta\|_{L^p} . \tag{2.21}$$

The following result is derived from Proposition 2.5.

Corollary 2.7 *Let $b > a > 0$ and Ω be an open domain of $\mathbb{R}^N$ such that $\Omega \subset B_b \setminus \overline{B_a}$. Then there exists a constant $C = C(s,p,N) > 0$ such that for any $\eta \in C_0^\infty(\Omega)$*

$$\|\eta\|_{\dot{B}^{s,p}} \leq \|\eta\|_{B^{s,p}} \leq C(b-a)^{sp} \|\eta\|_{\dot{B}^{s,p}} . \tag{2.22}$$

2.1.3 Heat potential and Besov space

If $\eta \in C_0^\infty(\Omega)$, we extend it by 0 outside Ω and set

$$\|\eta\|_{\tilde{B}^{s,p}} = \left(\iint_{Q_\infty} \left|t^{1-s/2}\partial_t \mathbb{H}[\eta]\right|^p dx\,\frac{dt}{t}\right)^{1/p} \tag{2.23}$$

It is well known (see e.g. [3]) that the Besov space $B^{s,p}(\Omega)$ can be defined directly as the space of $\eta \in L^p(\Omega)$ functions such that $\|\eta\|_{\dot{B}^{s,p}} < \infty$ or or such that $\|\eta\|_{\tilde{B}^{s,p}} < \infty$. It coincides with the the interpolation space $\left[W^{2,p}(\Omega), L^p(\Omega)\right]_{s/2,p}$ (see [24]). Furthermore, there exists $C = C(s,p,N) > 0$ such that

$$C^{-1}\left(\|\eta\|_{L^p} + \|\eta\|_{\dot{B}^{s,p}}\right) \leq \|\eta\|_{L^p} + \|\eta\|_{\tilde{B}^{s,p}} \leq C\left(\|\eta\|_{L^p} + \|\eta\|_{\dot{B}^{s,p}}\right) \quad \forall \eta \in B^{s,p}(\Omega). \tag{2.24}$$

Lemma 2.8 *Assume $0 < s < 1$ and $1 < p < \infty$ or $s = 1$ and $p = 2$. Then there exists a positive constant C, depending only on s, p, N, such that for any domain Ω, there holds*

$$C^{-1}\|\eta\|_{\dot{B}^{s,p}} \leq \|\eta\|_{\tilde{B}^{s,p}} \leq C\|\eta\|_{\dot{B}^{s,p}} \quad \forall \eta \in C_0^\infty(\Omega). \tag{2.25}$$

Proof. Let $\eta \in C_0^\infty(\mathbb{R}^N)$ and $\tau > 0$. Set $\eta_\tau(x) = \eta(\tau x)$, then (2.25) applied to η_τ yields to

$$C^{-1}\left(\|\eta\|_{L^p} + \tau^s\|\eta\|_{\dot{B}^{s,p}}\right) \leq \left(\|\eta\|_{L^p} + \tau^s\|\eta\|_{\tilde{B}^{s,p}}\right) \leq C\left(\|\eta\|_{L^p} + \tau^s\|\eta\|_{\dot{B}^{s,p}}\right).$$

Since it holds for any arbitrary large τ and $\eta \in C_0^\infty(\mathbb{R}^N)$, (2.25) follows. $\square$

We denote by $\mathcal{T}_\Omega(K)$ the set of functions $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on K. If Ω is a bounded subset of $\mathbb{R}^N$, we define the Besov capacity of a compact set $K \subset \Omega \subset \mathbb{R}^N$ by

$$R^\Omega_{s,p}(K) = \inf\{\|\eta\|^p_{\dot{B}^{s,p}} : \eta \in \mathcal{T}_\Omega(K)\}, \tag{2.26}$$

and the Bessel capacity relative to Ω by

$$C^\Omega_{s,p}(K) = \inf\{\|\eta\|^p_{B^{s,p}} : \eta \in \mathcal{T}_\Omega(K)\}. \tag{2.27}$$

We extend classicaly this capacity to any capacitable set $K \subset \Omega$. This capacity has the following scaling property.

Lemma 2.9 *For any $\tau > 0$ and any capacitable set $K \subset \Omega$, there holds*

$$R^{\Omega}_{s,p}(K) = \tau^{N-sp} R^{\tau^{-1}\Omega}_{s,p}(\tau^{-1}K). \tag{2.28}$$

Furthermore, if $\Omega \subset B_b \setminus \overline{B_a}$, there exists $c = c(b-a, b/a, N, s, p) > 0$ such that

$$c^{-1} C^{\Omega}_{s,p}(K) \leq R^{\Omega}_{s,p}(K) \leq c C^{\Omega}_{s,p}(K). \tag{2.29}$$

Finally, if $K \subset \Omega' \subset \overline{\Omega'} \subset \Omega$, there exists $c = c(N, s, p, \mathrm{dist}\,(\Omega', \Omega^c))$ such that

$$C_{s,p}(K) \leq C^{\Omega}_{s,p}(K) \leq c C_{s,p}(K). \tag{2.30}$$

Proof. The scaling property (2.28) is clear by change of variable. Estimate (2.29) is a consequence of Definition 2.6 and Proposition 2.5. For the last statement, the left-hand side is obvious. For the right-hand side, consider a smooth nonnegative cut-off function ζ which is 1 on $\overline{\Omega'}$, has value between 0 and 1 and has compact support in Ω. If $\eta \in \mathcal{T}_{\mathbb{R}^N}(K)$, $\zeta\eta \in \mathcal{T}_{\Omega}(K)$ and

$$\begin{aligned}
\|\zeta\eta\|^p_{B^{s,p}} &= \|\zeta\eta\|^p_{L^p(\Omega)} + \|\zeta\eta\|^p_{\dot{B}^{s,p}} \\
&\leq \|\eta\|^p_{L^p(\Omega)} + \|\eta\|^p_{\dot{B}^{s,p}} + \|\zeta\|^p_{\dot{B}^{s,\infty}} \|\eta\|^p_{L^p} \\
&\leq c \|\eta\|^p_{B^{s,p}} ,
\end{aligned}$$

where

$$\|\zeta\|_{\dot{B}^{s,\infty}} = \sup_{x \neq y} \frac{|\zeta(x) - \zeta(y)|}{|x-y|^s}$$

and $c \approx 1 + (\mathrm{dist}\,(\Omega', \Omega^c))^{-s}$. The proof follows. □

In the sequel we assume that $q \geq q_c$ and we take $p = q'$ and $s = 2/q$. If $K \subset \Omega$, Ω is bounded and $\eta \in \mathcal{T}_{\Omega}(K)$, we set

$$R[\eta] = |\nabla \mathbb{H}[\eta]|^2 + |\partial_t \mathbb{H}[\eta]| . \tag{2.31}$$

Lemma 2.10 *There exists $C = C(N, q) > 0$ such that for every $\eta \in \mathcal{T}_{\Omega}(K)$*

$$\|\eta\|^{q'}_{\tilde{B}^{2/q,q'}} \leq \iint_{Q_\infty} (R[\eta])^{q'} \, dx\, dt := \|R[\eta]\|^{q'}_{L^{q'}} \leq C \|\eta\|^{q'}_{\tilde{B}^{2/q,q'}} \tag{2.32}$$

Proof. Using (2.23) and Lemma 2.8, it follows from Corollary 2.7 that

$$\|\eta\|^{q'}_{\tilde{B}^{2/q,q'}} \approx \iint_{Q_\infty} |\partial_t \mathbb{H}[\eta]|^{q'} \, dxdt.$$

Using the Gagliardo-Nirenberg inequality in $\mathbb{R}^N$, an elementary elliptic estimate and the fact that $0 \leq \mathbb{H}[\eta] \leq 1$, we see that

$$\int_{\mathbb{R}^N} |\nabla(\mathbb{H}[\eta](.,t))|^{2q'} dx \leq C \left\|D^2 \mathbb{H}[\eta](.,t)\right\|^{q'}_{L^{q'}} \|\mathbb{H}[\eta](.,t)\|^{q'}_{L^\infty} \leq C \|\Delta \mathbb{H}[\eta](.,t)\|^{q'}_{L^{q'}} , \tag{2.33}$$

for all $t > 0$. Since $\partial_t \mathbb{H}[\eta] = \Delta \mathbb{H}[\eta]$, it implies (2.32). □

The dual space $B^{-2/q,q}(\Omega)$ of $B^{2/q,q'}(\Omega)$ is naturally endowed with the norm

$$\|\mu\|_{B^{-2/q,q}} = \sup\left\{\mu(\eta) : \eta \in B^{2/q,q'}(\Omega), \|\eta\|_{B^{2/q,q'}} \leq 1\right\}.$$

The following result is may be already known, but we have not found it in the literature. If μ is a bounded measure in $\mathbb{R}^N$, we denote by $\mathbb{H}[\mu]$ the solution of heat equation in Q_∞ with initial data μ.

Lemma 2.11 *Assume $q \geq q_c$. For any $T > 0$, there exist a constant $c > 0$ such that, for any bounded measure μ belonging to $B^{-2/q,q}(\mathbb{R}^N)$, there holds*

$$c^{-1}\|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}[\mu]\|_{L^q(Q_T)} \leq c\|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)}. \tag{2.34}$$

Furthermore, if $q > q_c$ there holds

$$c^{-1}\|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} \leq \|\mathbb{H}[\mu]\|_{L^q(Q_\infty)} \leq c\|\mu\|_{B^{-2/q,q}(\mathbb{R}^N)} + c\|\mu\|_{\mathfrak{M}(\mathbb{R}^N)}. \tag{2.35}$$

.

Proof. If $\mu \in B^{-2/q,q}(\mathbb{R}^N)$, there exists a unique $\omega \in B^{2-2/q,q}(\mathbb{R}^N)$ such that $\mu = (I-\Delta)\omega$, and $\|\mu\|_{B^{-2/q,q}} \approx \|\omega\|_{B^{2-2/q,q}}$. Applying standard interpolation methods to the analytic semi-group $e^{-t(I-\Delta)} = e^{-t}e^{t\Delta}$ (see e.g. [3], [41]) we obtain,

$$\begin{aligned}\left(\iint_{Q_\infty} \left|t^{1/q}(I-\Delta)\mathbb{H}[\omega]\right|^q dx\frac{e^{-qt}dt}{t}\right)^{1/q} &= \left(\iint_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t}\right)^{1/q} \\ &\approx \|\omega\|_{B^{2-2/q,q}} \\ &\approx \|\mu\|_{B^{-2/q,q}}.\end{aligned} \tag{2.36}$$

Clearly

$$e^{-qT}\iint_{Q_T} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t} \leq \iint_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t},$$

and

$$\begin{aligned}\iint_{Q_\infty} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t} &= \sum_{n=0}^{\infty}\iint_{Q_{T+n+1}\setminus Q_{T+n}} \left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{e^{-qt}dt}{t} \\ &= \sum_{n=0}^{\infty}\iint_{Q_T} |\mathbb{H}[\mu](s+n)|^q e^{-q(s+n)}ds \\ &\leq \left(\sum_{n=0}^{\infty} e^{-qn}\right)\iint_{Q_T} \left|t^{1/q}\mathbb{H}[\mu]\right|^q \frac{dt}{t}.\end{aligned}$$

This implies (2.34). Furthermore, $\||\mathbb{H}[\mu](.,t)|\|^q_{L^q} \leq ct^{-N(q-1)/2}\|\mu\|^q_{\mathfrak{M}}$, thus $\mathbb{H}[\mu] \in L^q(Q_\infty)$ if $q > q_c$ (but this does not hold if $q = q_c$). If $q > q_c$ (equivalently $N(q-1)/2 > 1$),

$$\begin{aligned}\iint_{Q_\infty}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t}&=\sum_{n=0}^{\infty}\iint_{Q_{T+n+1}\setminus Q_{T+n}}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t}\\&=\iint_{Q_T}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t}+\iint_{Q_T}\sum_{n=1}^{\infty}|\mathbb{H}[\mu](s+n)|^q\,dxds\\&\leq\iint_{Q_T}\left|t^{1/q}\mathbb{H}[\mu]\right|^q dx\frac{dt}{t}+C\left(\sum_{n=1}^{\infty}n^{-N(q-1)/2}\right)\|\mu\|^q_{\mathfrak{M}}.\end{aligned}$$

Thus we obtain (2.35). □

2.2 Global L^q-estimates

Let $\rho>0$, we assume (2.3) holds. With the previous notations, $\mathcal{T}_{r,r+\rho}(K)$ denotes the set of functions $\eta\in C_0^\infty(B_{r+\rho})$, such that $0\leq\eta\leq1$ and value 1 on K. If $\eta\in\mathcal{T}_{r,\rho}(K)$, we set

$$\eta^*=1-\eta\ \text{ and }\ \zeta=\mathbb{H}[\eta^*]^{2q'}.$$

Lemma 2.12 *Assume u is a positive solution of* (2.1) *in Q_∞. There exists $C=C(N,q)>0$ such that for every $T>0$ and every compact set $K\subset B_r$,*

$$\iint_{Q_T}u^q\zeta dx\,dt+\int_{\mathbb{R}^N}(u\zeta)(x,T)dx\leq C\|R[\eta]\|^{q'}_{L^{q'}}\qquad\forall\eta\in\mathcal{T}_{r,\rho}(K).\tag{2.37}$$

Proof. We recall that there always holds

$$0\leq u(x,t)\leq\left(\frac{1}{t(q-1)}\right)^{\frac{1}{q-1}}\qquad\forall(x,t)\in Q_\infty,\tag{2.38}$$

and

$$0\leq u(x,t)\leq\left(\frac{C}{t+(|x|-r)^2}\right)^{\frac{1}{q-1}}\qquad\forall(x,t)\in Q_\infty\setminus B_r\times\mathbb{R},\tag{2.39}$$

by the Brezis-Friedman estimate [7]. Since η^* vanishes in an open neighborhood $\mathcal{N}_1$, for any open subset $\mathcal{N}_2$ such that $K\subset\mathcal{N}_2\subset\overline{\mathcal{N}}_2\subset\mathcal{N}_1$ there exist $c_2=c_{\mathcal{N}_2}>0$ and $C_2=C_{\mathcal{N}_2}>0$ such that

$$\mathbb{H}[\eta^*](x,t)\leq C_2e^{-\frac{c_2}{t}},\qquad\forall(x,t)\in Q_T^{\mathcal{N}_2}.$$

Therefore

$$\lim_{t\to0}\int_{\mathbb{R}^N}(u\zeta)(x,t)dx=0.$$

Thus ζ is an admissible test function and one has

$$\iint_{Q_T}u^q\zeta dx\,dt+\int_{\mathbb{R}^N}(u\zeta)(x,T)dx=\iint_{Q_T}u(\partial_t\zeta+\Delta\zeta)dx\,dt.\tag{2.40}$$

Notice that the two terms on the left-hand side are nonnegative. Put $\mathbb{H}_{\eta^*} = \mathbb{H}[\eta^*]$, then

$$\begin{aligned}\partial_t\zeta + \Delta\zeta &= 2q'\mathbb{H}_{\eta^*}^{2q'-1}\left(\partial_t\mathbb{H}_{\eta^*} + \Delta\mathbb{H}_{\eta^*}\right) + 2q'(2q'-1)\mathbb{H}_{\eta^*}^{2q'-2}|\nabla\mathbb{H}_{\eta^*}|^2,\\ &= 2q'\mathbb{H}_{\eta^*}^{2q'-1}\left(\partial_t\mathbb{H}_{\eta} + \Delta\mathbb{H}_{\eta}\right) + 2q'(2q'-1)\mathbb{H}_{\eta}^{2q'-2}|\nabla\mathbb{H}_{\eta}|^2,\end{aligned}$$

because $\mathbb{H}_{\eta^*} = 1 - \mathbb{H}_\eta$, hence

$$u(\partial_t\zeta + \Delta\zeta) = u\mathbb{H}_{\eta^*}^{2q'/q}\left[2q'(2q'-1)\mathbb{H}_{\eta^*}^{2q'-2-2q'/q}|\nabla\mathbb{H}_\eta|^2 - 2q'\mathbb{H}_{\eta^*}^{2q'-1-2q'/q}(\Delta\mathbb{H}_\eta + \partial_t\mathbb{H}_\eta)\right].$$

Finally, since $2q' - 2 - 2q'/q = 0$ and $0 \leq \mathbb{H}_{\eta^*} \leq 1$, there holds

$$\left|\iint_{Q_T} u(\partial_t\zeta + \Delta\zeta)dx\,dt\right| \leq C(q)\left(\iint_{Q_T} u^q\zeta dx\,dt\right)^{1/q}\left(\iint_{Q_T} R^{q'}(\eta)dx\,dt\right)^{1/q'},$$

where

$$R(\eta) = |\nabla\mathbb{H}_\eta|^2 + |\Delta\mathbb{H}_\eta + \partial_t\mathbb{H}_\eta|.$$

Using Lemma 2.10 one obtains (2.37). □

Proposition 2.13 *Under the assumptions of Lemma 2.12, let $r > 0$, $\rho > 0$, $T \geq (r+\rho)^2$*

$$\mathcal{E}_{r+\rho} := \{(x,t) : |x|^2 + t \leq (r+\rho)^2\}$$

and $Q_{r+\rho,T} = Q_T \setminus \mathcal{E}_{r+\rho}$. There exists $C = C(N,q,T) > 0$ such that

$$\iint_{Q_{r+\rho,T}} u^q dx\,dt + \int_{\mathbb{R}^N} u(x,T)dx \leq C\|R[\eta]\|_{L^{q'}}^{q'} \qquad \forall \eta \in \mathcal{T}_{r,\rho}(K). \tag{2.41}$$

Proof. In view of Lemma 2.12 we only have to show that there exists a positive constant $c(N,q)$ such that, for η as above and $T \geq (r+\rho)^2$,

$$\zeta = \mathbb{H}\eta^{*2q'} > c(N,q).$$

Since, by assumption $K \subset B_r$, $\eta^* \equiv 1$ outside $B_{r+\rho}$ and $0 \leq \eta^* \leq 1$,

$$\begin{aligned}\mathbb{H}[\eta^*](x,t) \geq \mathbb{H}[1 - \chi_{B_{r+\rho}}](x,t) &= \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\geq r+\rho} e^{-\frac{|x-y|^2}{4t}}dy,\\ &= 1 - \left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\leq r+\rho} e^{-\frac{|x-y|^2}{4t}}dy.\end{aligned}$$

For $(x,t) \in Q_{r+\rho,T}$, put $x = (r+\rho)\xi$, $y = (r+\rho)\upsilon$ and $t = (r+\rho)^2\tau$. Then $(\xi,\tau) \in Q_{1,\frac{T}{(r+\rho)^2}}$ and

$$\left(\frac{1}{4\pi t}\right)^{\frac{N}{2}}\int_{|y|\leq r+\rho} e^{-\frac{|x-y|^2}{4t}}dy = \left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\leq 1} e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon.$$

We claim that

$$\max\left\{\left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\leq 1}e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon:(\xi,\tau)\in Q_{1,\frac{T}{(r+\rho)^2}}\right\}=\ell, \tag{2.42}$$

for some $\ell=\ell(N,\frac{T}{(r+\rho)^2})\in(0,1]$, and ℓ is actually independent of $\frac{T}{(r+\rho)^2}$if this quantity is larger than 1. We recall that

$$\left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}}\int_{|\upsilon|\leq 1}e^{-\frac{|\xi-\upsilon|^2}{4\tau}}d\upsilon<1 \qquad \forall\tau>0. \tag{2.43}$$

If the maximum is achieved for some $(\bar{\xi},\bar{\tau})\in Q_{1,\frac{T}{(r+\rho)^2}}$, it is smaller than 1 and

$$\mathbb{H}[\eta^*](x,t)\geq\mathbb{H}[1-\chi_{B_{r+\rho}}](x,t)\geq 1-\ell>0, \qquad \forall(x,t)\in Q_{r+\rho,T}. \tag{2.44}$$

Let us assume that the maximum is achieved following a sequence $\{(\xi_n,\tau_n)\}$ with $\tau_n\to 0$ and $|\xi_n|\to\alpha\geq 1$. Then

$$\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}}\int_{|\upsilon|\leq 1}e^{-\frac{|\xi_n-\upsilon|^2}{4\tau_n}}d\upsilon=\left(\frac{1}{4\pi\tau_n}\right)^{\frac{N}{2}}\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon\leq\frac{1}{2}.$$

To verify this, note that $B_1(\xi_n)\cap B_1(-\xi_n)=\emptyset$, so that

$$\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon+\int_{B_1(-\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon<\int_{\mathbb{R}^N}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon<1$$

and

$$\int_{B_1(\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon=\int_{B_1(-\xi_n)}e^{-\frac{|\upsilon|^2}{4\tau_n}}d\upsilon.$$

If the supremum is achieved with a sequence $\{(\xi_n,\tau_n)\}$ such that $|\xi_n|\to\infty$, the same argument applies. Finally if $\{\xi_n\}$ is bounded but $\tau_n\to\infty$ then the expression in (2.43) tends to zero. Therefore (2.43) holds. Put $C=(1-\ell)^{-1}$, then

$$\iint_{Q_{r,T}}u^q dx\,dt+\int_{\mathbb{R}^N}u(.,T)dx\leq C\left\|R[\eta]\right\|_{L^{q'}}^{q'}, \tag{2.45}$$

and (2.41) follows. □

2.3 Pointwise estimates

In this subsection u is a positive solution of (2.1) in Q_∞ and the assumptions of Lemma 2.12 hold. We first derive a rough pointwise estimate.

Lemma 2.14 *There exists a constant $C=C(N,q)>0$ such that, for any $\eta\in\mathcal{T}_{r,\rho}(K)$,*

$$u(x,(r+2\rho)^2)\leq\frac{C\left\|R[\eta]\right\|_{L^{q'}}^{q'}}{(\rho(r+\rho))^{\frac{N}{2}}}, \qquad \forall x\in\mathbb{R}^N. \tag{2.46}$$

Proof. We recall that

$$\int_s^T \int_{\mathbb{R}^N} u^q dx\,dt + \int_{\mathbb{R}^N} u(x,T)dx = \int_{\mathbb{R}^N} u(x,s)dx \qquad \forall T > s > 0, \tag{2.47}$$

and

$$\int_{\mathbb{R}^N} u(.,s)dx \leq C\,\|R[\eta]\|_{L^{q'}}^{q'} \qquad \forall T > s \geq (r+\rho)^2, \tag{2.48}$$

by Proposition 2.13. Using the fact that

$$u(x,\tau+s) \leq \mathbb{H}[u(.,s)](x,\tau) \leq \left(\frac{1}{4\pi\tau}\right)^{\frac{N}{2}} \int_{\mathbb{R}^N} u(.,s)dx,$$

(2.46) follows from (2.48) with $s = (r+\rho)^2$ and $\tau = (r+2\rho)^2 - (r+\rho)^2 \approx \rho(r+\rho)$. $\square$

The above estimate does not take into account the fact that $u(x,0) = 0$ if $|x| \geq r$. It is mainly interesting if $|x| \leq r$. In order to derive a sharper estimate which takes this fact into account, we need some lateral boundary estimates.

Lemma 2.15 *Let $\gamma \geq r + 2\rho$ and $c > 0$ and either $N = 1$ or 2 and $0 \leq t \leq c\gamma^2$ for some $c > 0$, or $N \geq 3$ and $t > 0$. Then, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$\int_0^t \int_{\partial B_\gamma} udSd\tau \leq C_5\gamma\,\|R[\eta]\|_{L^{q'}}^{q'}. \tag{2.49}$$

where $C > 0$ depends on N, q and c if $N = 1, 2$ or depends only on N and q if $N \geq 3$.

Proof. First we assume $N = 1$ or 2. Put $G^\gamma := B_\gamma^c \times (-\infty, 0)$ and $\partial_\ell G^\gamma = \partial B_\gamma \times (-\infty, 0)$. We set

$$h_\gamma(x) = 1 - \frac{\gamma}{|x|},$$

and let ψ_γ be the solution of

$$\begin{aligned} \partial_\tau\psi_\gamma + \Delta\psi_\gamma = 0 \qquad & \text{in } G^\gamma, \\ \psi_\gamma = 0 \qquad & \text{on } \partial_\ell G^\gamma, \\ \psi_\gamma(.,0) = h_\gamma \qquad & \text{in } B_\gamma^c. \end{aligned} \tag{2.50}$$

Thus the function

$$\tilde{\psi}(x,\tau) = \psi_\gamma(\gamma x, \gamma^2\tau)$$

satisfies

$$\begin{aligned} \partial_t\tilde{\psi} + \Delta\tilde{\psi} = 0 \qquad & \text{in } G^1 \\ \tilde{\psi} = 0 \qquad & \text{on } \partial_\ell G^1 \\ \tilde{\psi}(.,0) = \tilde{h} \qquad & \text{in } B_1^c, \end{aligned} \tag{2.51}$$

and $\tilde{h}(x) = 1 - |x|^{-1}$. By the maximum principle $0 \leq \tilde{\psi} \leq 1$, and by Hopf Lemma

$$-\frac{\partial\tilde{\psi}}{\partial\mathbf{n}}\Big|_{\partial B_1\times[-c,0]} \geq \theta > 0, \tag{2.52}$$

where $\theta = \theta(N, c)$. Then $0 \leq \psi_\gamma \leq 1$ and

$$-\frac{\partial \psi_\gamma}{\partial \mathbf{n}}\Big|_{\partial B_\gamma \times [-\gamma^2, 0]} \geq \theta/\gamma. \tag{2.53}$$

Multiplying (1.1) by $\psi_\gamma(x, \tau - t) = \psi^*_\gamma(x, \tau)$ and integrating on $B^c_\gamma \times (0, t)$ yields to

$$\int_0^t \int_{B^c_\gamma} u^q \psi^*_r dx d\tau + \int_{B^c_\gamma} (u h_\gamma)(x, t) dx - \int_0^t \int_{\partial B_\gamma} \frac{\partial u}{\partial \mathbf{n}} \psi^*_\gamma dS d\tau = -\int_0^t \int_{\partial B_\gamma} \frac{\partial \psi^*_\gamma}{\partial \mathbf{n}} u d\sigma d\tau. \tag{2.54}$$

Since ψ^*_γ is bounded from above by 1, estimate (2.49) follows from (2.53) and Proposition 2.13 (notice that $B^c_\gamma \times (0, t) \subset \mathcal{E}^c_\gamma$), first by taking $t = T = \gamma^2 \geq (r + 2\rho)^2$, and then for any $t \leq \gamma^2$.

If $N \geq 3$, we proceed as above except that we take

$$h_\gamma(x) = 1 - \left(\frac{\gamma}{|x|}\right)^{N-2}.$$

Then $\psi_\gamma(x, t) = h_\gamma(x)$ and $\theta = N - 2$ is independent of the length of the time interval. This leads to the conclusion. $\square$

Lemma 2.16 *I- Let M, $a > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that*

$$0 \leq \eta(x) \leq M e^{-a|x|^2} \qquad a.e.\ in\ \mathbb{R}^N. \tag{2.55}$$

Then, for any $t > 0$,

$$0 \leq \mathbb{H}[\eta](x, t) \leq \frac{M}{(4at + 1)^{\frac{N}{2}}} e^{-\frac{a|x|^2}{4at+1}} \qquad \forall x \in \mathbb{R}^N. \tag{2.56}$$

II- Let M, a, $b > 0$ and $\eta \in L^\infty(\mathbb{R}^N)$ such that

$$0 \leq \eta(x) \leq M e^{-a(|x|-b)^2_+} \qquad a.e.\ in\ \mathbb{R}^N. \tag{2.57}$$

Then, for any $t > 0$,

$$0 \leq \mathbb{H}[\eta](x, t) \leq \frac{M e^{-\frac{a(|x|-b)^2_+}{4at+1}}}{(4at + 1)^{\frac{N}{2}}} \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0. \tag{2.58}$$

Proof. For the first statement, put $a = \frac{1}{4}s$. Then

$$0 \leq \eta(x) \leq M(4\pi s)^{\frac{N}{2}} \frac{1}{(4\pi s)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4s}} = C(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, s).$$

By the order property of the heat kernel,

$$0 \leq \mathbb{H}[\eta](x, t) \leq M(4\pi s)^{\frac{N}{2}} \mathbb{H}[\delta_0](x, t + s) = M \left(\frac{s}{t + s}\right)^{\frac{N}{2}} e^{-\frac{|x|^2}{4(t+s)}},$$

and (2.56) follows by replacing s by $\frac{1}{4}a$.

For the second statement, let $\tilde{a} < a$ and $R = \max\{e^{-a(r-b)_+^2 + \tilde{a}r^2} : r \geq 0\}$. A direct computation gives $R = e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}$, and (2.58) implies

$$0 \leq \eta(x) \leq M e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}} e^{-\tilde{a}|x|^2}.$$

Applying the statement I, we derive

$$0 \leq \mathbb{H}[\eta](x,t) \leq \frac{C e^{\frac{a\tilde{a}b^2}{a-\tilde{a}}}}{(4\tilde{a}t+1)^{\frac{N}{2}}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \qquad \forall x \in \mathbb{R}^N,\ \forall t > 0. \tag{2.59}$$

Since for any $x \in \mathbb{R}^N$ and $t > 0$,

$$(4\tilde{a}t+1)^{-\frac{N}{2}} e^{-\frac{\tilde{a}|x|^2}{4\tilde{a}t+1}} \leq e^{-\frac{a\tilde{a}b^2}{a-\tilde{a}}} (4at+1)^{-\frac{N}{2}} e^{-\frac{a(|x|-b)^2}{4at+1}},$$

(2.58) follows from (2.59). $\square$

Lemma 2.17 *There exists a constant $C = C(N,q) > 0$ such that, for any $\eta \in \mathcal{T}_{r,\rho}(K)$, there holds*

$$u(x,(r+2\rho)^2) \leq C \max\left\{\frac{r+\rho}{(|x|-r-2\rho)^{N+1}}, \frac{|x|-r-2\rho}{(r+\rho)^{N+1}}\right\} e^{-\frac{(|x|-(r+2\rho))^2}{4(r+2\rho)^2}} \|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.60}$$

for any $x \in \mathbb{R}^N \setminus B_{r+3\rho}$.

Proof. It is classical that the Dirichlet heat kernel $H^{B_1^c}$ in the complement of B_1 satisfies, for some $C = C(N) > 0$,

$$H^{B_1^c}(x',y',t',s') \leq C_7(t'-s')^{-(N+2)/2}(|x'|-1)e^{-\frac{|x'-y'|^2}{4(t'-s')}}, \tag{2.61}$$

for $t' > s'$. By performing the change of variable $x' \mapsto (r+2\rho)x'$, $t' \mapsto (r+2\rho)^2 t'$, for any $x \in \mathbb{R}^N \setminus B_{r+2\rho}$ and $0 \leq t \leq T$, one obtains

$$u(x,t) \leq C(|x|-r-2\rho) \int_0^t \int_{\partial B_{r+2\rho}} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{1+\frac{N}{2}}} u(y,s)d\sigma(y)ds. \tag{2.62}$$

The right-hand side term in (2.62) is smaller than

$$\max\left\{\frac{C(|x|-r-2\rho)}{(t-s)^{1+\frac{N}{2}}} e^{-\frac{(|x|-r-2\rho)^2}{4(t-s)}} : s \in (0,t)\right\} \int_0^t \int_{\partial B_{r+2\rho}} u(y,s)d\sigma(y)ds.$$

We fix $t = (r+2\rho)^2$ and $|x| \geq r+3\rho$. Since

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : s \in \big(0,(r+2\rho)^2\big)\right\}$$
$$= (|x|-r-2\rho)^{-2-N} \max\left\{\frac{e^{-\frac{1}{4\sigma}}}{\sigma^{1+\frac{N}{2}}} : 0 < \sigma < \left(\frac{r+2\rho}{|x|-r-2\rho}\right)^2\right\},$$

a direct computation gives

$$\max\left\{\frac{e^{-\frac{1}{4}\sigma}}{\sigma^{1+\frac{N}{2}}}:0<\sigma<\left(\frac{r+2\rho}{|x|-r-2\rho}\right)^2\right\}$$

$$=\begin{cases}(2N+4)^{1+\frac{N}{2}}e^{-(N+2)/2} & \text{if } r+3\rho\le|x|\le(r+2\rho)(1+\sqrt{4+2N}),\\ \left(\dfrac{|x|-r-2\rho}{r+2\rho}\right)^{2+N}e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2} & \text{if } |x|\ge(r+2\rho)(1+\sqrt{4+2N}).\end{cases}$$

Thus there exists a constant $C(N)>0$ such that

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}}:s\in\big(0,(r+2\rho)^2\big)\right\}\le C(N)\rho^{-2-N}e^{-\left(\frac{|x|-(r+2\rho)}{2r+4\rho}\right)^2}. \tag{2.63}$$

Combining this estimate with (2.49) with $\gamma=r+2\rho$ and (2.62), one derives (2.60). □

Lemma 2.18 *There exists a constant $C=C(N,q)>0$ such that*

$$0\le u(x,(r+2\rho)^2)\le C\max\left\{\frac{(r+\rho)^3}{\rho(|x|-r-2\rho)^{N+1}},\frac{1}{(r+\rho)^{N-1}\rho}\right\}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2}\|R[\eta]\|^{q'}_{L^{q'}}, \tag{2.64}$$

for every $x\in\mathbb{R}^N\setminus B_{r+3\rho}$.

Proof. This is a direct consequence of the inequality

$$(|x|-r-2\rho)e^{-\left(\frac{|x|-r-2\rho}{2r+4\rho}\right)^2}\le\frac{C(r+\rho)^2}{\rho}e^{-\left(\frac{|x|-r-3\rho}{2r+4\rho}\right)^2},\qquad\forall x\in B^c_{r+2\rho}, \tag{2.65}$$

and Lemma 2.17. □

Lemma 2.19 *There exists a constant $C=C(N,q)>0$ such that, for any $\eta\in\mathcal{T}_{r,\rho}(K)$, the following estimate holds*

$$u(x,t)\le\frac{C\tilde{M}e^{-\frac{(|x|-r-3\rho)_+^2}{4t}}}{t^{\frac{N}{2}}}\|R[\eta]\|^{q'}_{L^{q'}},\qquad\forall x\in\mathbb{R}^N,\ \forall t\ge(r+2\rho)^2, \tag{2.66}$$

where

$$\tilde{M}=\tilde{M}(x,r,\rho)=\begin{cases}\left(1+\frac{r}{\rho}\right)^{\frac{N}{2}} & \textit{if } |x|<r+3\rho\\ \frac{(r+\rho)^{N+3}}{\rho(|x|-r-2\rho)^{N+2}} & \textit{if } r+3\rho\le|x|\le c^*_N(r+2\rho)\\ 1+\frac{r}{\rho} & \textit{if } |x|\ge c^*_N(r+2\rho)\end{cases} \tag{2.67}$$

with $c^*_N=1+\sqrt{4+2N}$.

Proof. It follows by the maximum principle

$$u(x,t) \le \mathbb{H}[u(.,(r+2\rho)^2)](x,t-(r+2\rho)^2).$$

for $t \ge (r+2\rho)^2$ and $x \in \mathbb{R}^N$. By Lemma 2.14 and Lemma 2.18

$$u(x,(r+2\rho)^2) \le C_{10}\tilde{M}e^{-\frac{(|x|-r-3\rho)^2}{4(r+2\rho)^2}} \, \|R[\eta]\|_{L^{q'}}^{q'},$$

where

$$\tilde{M} = \begin{cases} ((r+\rho)\rho)^{-\frac{N}{2}} & \text{if } |x| < r+3\rho \\ \frac{(r+\rho)^3}{\rho}\left(|x|-r-2\rho)\right)^{N+2} & \text{if } r+3\rho \le |x| \le c_N^*(r+2\rho) \\ \frac{1}{(r+\rho)^{N-1}\rho} & \text{if } |x| \ge c_N^*(r+2\rho) \end{cases}$$

Applying Lemma 2.16 with $a = (2r+4\rho)^{-2}$, $b = r+3\rho$ and t replaced by $t-(r+2\rho)^2$ implies

$$u(x,t) \le C\frac{(r+2\rho)^N\tilde{M}}{t^{\frac{N}{2}}}e^{-\frac{(|x|-r-3\rho)^2}{4t}} \, \|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.68}$$

for all $x \in B^c_{r+3\rho}$ and $t \ge (r+2\rho)^2$, which is (2.66). □

The next estimate gives a precise upper bound for u when t is not bounded from below.

Lemma 2.20 *Assume that* $0 < t \le (r+2\rho)^2$, *then there exists a constant* $C = C(N,q) > 0$ *such that the following estimate holds*

$$u(x,t) \le C(r+\rho)\max\left\{\frac{1}{(|x|-r-2\rho)^{N+1}}, \frac{1}{\rho t^{\frac{N}{2}}}\right\}e^{-\frac{(|x|-r-3\rho)^2}{4t}} \, \|R[\eta]\|_{L^{q'}}^{q'}, \tag{2.69}$$

for any $(x,t) \in \mathbb{R}^N \setminus B_{r+3\rho} \times (0,(r+2\rho)^2]$.

Proof. Thanks to (2.49) the following estimate is a straightforward variant of (2.60) for any $|x| \ge r+2\rho$,

$$u(x,t) \le C_8(|x|-r-2\rho)(r+2\rho)\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \le t\right\}\|R[\eta]\|_{L^{q'}}^{q'}. \tag{2.70}$$

Clearly

$$\max\left\{\frac{e^{-\frac{(|x|-r-2\rho)^2}{4s}}}{s^{1+\frac{N}{2}}} : 0 < s \le t\right\} = \begin{cases} (2N+4)^{1+\frac{N}{2}}(|x|-r-2\rho)^{-N-2}e^{-\frac{N+2}{2}} & \text{if } 0 < |x| \le r+2\rho+\sqrt{2t(N+2)} \\ \frac{e^{-\frac{(|x|-r-2\rho)^2}{4t}}}{t^{1+\frac{N}{2}}} & \text{if } |x| > r+2\rho+\sqrt{2t(N+2)}. \end{cases}$$

By elementary analysis, if $x \in B^c_{r+3\rho}$,

$$(|x|-r-2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq e^{-\frac{(|x|-r-3\rho)^2}{4t}} \begin{cases} \rho e^{-\frac{\rho^2}{4t}} & \text{if } 2t < \rho^2 \\ \dfrac{2t}{\rho} e^{-1+\frac{\rho^2}{4t}} & \text{if } \rho^2 \leq 2t \leq 2(r+2\rho)^2. \end{cases}$$

However, since

$$\frac{\rho}{t} e^{-\frac{\rho^2}{4t}} \leq \frac{4}{\rho},$$

we derive

$$(|x|-r-2\rho)e^{-\frac{(|x|-r-2\rho)^2}{4t}} \leq \frac{Ct}{\rho} e^{-\frac{(|x|-r-3\rho)^2}{4t}},$$

and (2.69) follows. $\square$

Remark. In the subcritical case $1 < q < q_c$, it is easy to show by using Lemma 2.20, that any positive solution u of (2.1), such that $u(x,0) = 0$ for $x \neq 0$, satisfies

$$u(x,t) \leq Ct^{-\frac{1}{q-1}} \min\left\{1, \left(\frac{|x|}{\sqrt{t}}\right)^{\frac{2}{q-1}-N} e^{-\frac{|x|^2}{4t}}\right\} \qquad \forall (x,t) \in Q_\infty. \tag{2.71}$$

This upper estimate corresponds to the one obtained in [8]. If $F = \overline{B}_r$ the upper estimate is less esthetic. However, it is proved in [28] by a barrier method that, if the initial trace of positive solution u of (2.1), vanishes outside F, and if $1 < q < 3$, there holds

$$u(x,t) \leq t^{-\frac{1}{q-1}} f_1((|x|-r)/\sqrt{t}) \qquad \forall (x,t) \in Q_\infty,\ |x| \geq r, \tag{2.72}$$

where $f = f_1$ is the unique positive (and radial) solution of

$$\begin{cases} f'' + \dfrac{y}{2} f' + \dfrac{1}{q-1} f - f^q = 0 \quad \text{in } (0,\infty) \\ f'(0) = 0\,,\ \lim_{y\to\infty} |y|^{\frac{2}{q-1}} f(y) = 0. \end{cases} \tag{2.73}$$

Notice that the existence of f_1 follows from [8] since q belongs to the subcritical range on exponents in dimension one. Furthermore f_1 has the following asymptotic expansion

$$f_1(y) = Cy^{(3-q)/(q-1)} e^{-y^2/4t}(1 + \circ(1))) \quad \text{as } y \to \infty.$$

2.4 The upper Wiener test

Definition 2.21 We define on $\mathbb{R}^N \times \mathbb{R}$ the two *parabolic distances* δ_2 and δ_∞ by

$$\delta_2[(x,t),(y,s)] := \sqrt{|x-y|^2 + |t-s|}, \tag{2.74}$$

and

$$\delta_\infty[(x,t),(y,s)] := \max\{|x-y|, \sqrt{|t-s|}\}. \tag{2.75}$$

If $K\subset\mathbb{R}^N$ and $i=2,\infty$,

$$\delta_i[(x,t),K]=\inf\{\delta_i[(x,t),(y,0)]:y\in K\}=\left\{\begin{array}{ll}\max\left\{\operatorname{dist}(x,K),\sqrt{|t|}\right\} & \text{if } i=\infty,\\ \sqrt{\operatorname{dist}^2(x,K)+|t|} & \text{if } i=2.\end{array}\right.$$

For $\beta>0$ and $i=2,\infty$, we denote by $\mathcal{B}^i_\beta(m)$ the parabolic ball of center $m=(x,t)$ and radius β in the parabolic distance δ_i.

Let K be *any* compact subset of $\mathbb{R}^N$ and $\overline{u}_K$ the maximal solution of (1.1) which blows up on K. The function $\overline{u}_K$ is constructed in [28] as being the decreasing limit of the $\overline{u}_{K_\epsilon}$ ($\epsilon>0$) when $\epsilon\to 0$, where

$$K_\epsilon=\{x\in\mathbb{R}^N:\operatorname{dist}(x,K)\leq\epsilon\}$$

and $\overline{u}_{K_\epsilon}=\lim_{k\to\infty}u_{k,K_\epsilon}=\overline{u}_K$, where u_k is the solution of the classical problem,

$$\left\{\begin{array}{rll}\partial_t u_k-\Delta u_k+u_k^q=0 & & \text{in } Q_T,\\ u_k=0 & & \text{on } \partial_\ell Q_T,\\ u_k(.,0)=k\chi_{K_\epsilon} & & \text{in } \mathbb{R}^N.\end{array}\right. \tag{2.76}$$

If $(x,t)=m\in\mathbb{R}^N\times(0,T]$, we set $d_K=\operatorname{dist}(x,K)$, $D_K=\max\{|x-y|:y\in K\}$ and $\lambda=\sqrt{d_K^2+t}=\delta_2[m,K]$. We define a slicing of K, by setting $d_n=d_n(K,t):=\sqrt{nt}$ ($n\in\mathbb{N}$), $d_n^{\pm}=\left(\sqrt{nt}\pm\dfrac{\sqrt{t}}{\sqrt{n}}\right)_+$ (the positive part is only needed when $n=0$) and

$$T_n^*=\overline{B}_{d^+_{n+1}}(x)\setminus B_{d^-_n}(x)\,,\ T_n=\overline{B}_{d_{n+1}}(x)\setminus B_{d_n}(x),\qquad\forall n\in\mathbb{N},$$

thus $T_0^*=B_{2\sqrt{t}}(x)$, $T_0=B_{\sqrt{t}}(x)$, and

$$K_n(x,t)=K\cap T_n(x,t)\ \text{ for } n\in\mathbb{N} \text{ and } \mathcal{Q}_n(x,t)=K\cap B_{d_{n+1}}(x,t).$$

When there is no ambiguity, we will skip the (x,t) variable in the above sets. The main result of this section is the following discrete upper Wiener-type estimate.

Theorem 2.22 *Assume $q\geq q_c$. Then there exists $C=C(N,q,T)>0$ such that*

$$\overline{u}_K(x,t)\leq\frac{C}{t^{\frac{N}{2}}}\sum_{n=0}^{a_t}d_{n+1}^{N-\frac{2}{q-1}}e^{-\frac{n}{4}}C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right)\qquad\forall(x,t)\in Q_T, \tag{2.77}$$

where a_t is the largest integer j such that $K_j\neq\emptyset$.

With no loss of generality, we can assume that $x=0$. Furthermore, in considering the scaling transformation $u_\ell(y,t)=\ell^{\frac{1}{q-1}}u(\sqrt{\ell}y,\ell t)$, with $\ell>0$, we can assume $t=1$. Thus the new compact singular set of the initial trace becomes $K/\sqrt{\ell}$, that we still denote K. We also set $a_K=a_{K,1}$ For $n\in\mathbb{N}_*$ set $\delta_n=d_{n+1}-d_n$, then $\frac{1}{2\sqrt{n+1}}\leq\delta_n\leq\frac{1}{2\sqrt{n}}$. By convention $\delta_0=1$. It is possible to exhibit a collection Θ_n of points $a_{n,j}$ with center on the sphere $\Sigma_n = \{y \in \mathbb{R}^N : |y| = (d_{n+1} + d_n)/2\}$, such that

$$T_n \subset \bigcup_{a_{n,j} \in \Theta_n} B_{\delta_n}(a_{n,j}), \quad |a_{n,j} - a_{n,k}| \geq \delta_n \text{ and } \#\Theta_n \leq Cn^{N-1},$$

for some constant $C = C(N)$. If $K_{n,j} = K_n \cap B_{\delta_n}(a_{n,j})$, there holds

$$K = \bigcup_{0 \leq n \leq a_K} \bigcup_{a_{n,j} \in \Theta_n} K_{n,j}.$$

The first intermediate step is based on the *quasi-additivity* property of capacities developed in [2].

Lemma 2.23 *Let $q \geq q_c$. There exists a constant $C = C(N,q)$ such that*

$$\sum_{a_{n,j} \in \Theta_n} R_{2/q,q'}^{B_{2\delta_n}(a_{n,j})}(K_{n,j}) \leq C d_{n+1}^{N - \frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \qquad \forall n \in \mathbb{N}_*. \tag{2.78}$$

Proof. The following result is proved in [2, Th 3]: if the spheres $B_{\rho_j^\theta}(b_j)$, $\theta = 1 - 2/N(q-1)$, are disjoint in $\mathbb{R}^N$ and G is an analytic subset of $\bigcup B_{\rho_j}(b_j)$ where the ρ_j are positive and smaller than some $\rho^* > 0$, there holds

$$C_{2/q,q'}(G) \leq \sum_j C_{2/q,q'}(G \cap B_{\rho_j}(b_j)) \leq AC_{2/q,q'}(G), \tag{2.79}$$

for some A depending on N, q and ρ^*. This property is called *quasi-additivity.* We define for $n \in \mathbb{N}_*$,

$$\tilde{T}_n = d_{n+1} T_n, \quad \tilde{K}_n = d_{n+1} K_n \quad \text{and} \quad \tilde{\mathcal{Q}}_n = d_{n+1} \mathcal{Q}_n.$$

Since $K_{n,j} \subset B_{\delta_n}(a_{n,j})$, it follows that

$$\tilde{K}_{n,j} := d_{n+1} K_{n,j} \subset B_{d_{n+1}\delta_n}(\tilde{a}_{n,j}).$$

Note that by Lemma 2.9

$$\begin{aligned} R_{2/q,q'}^{B_{2\delta_n}(a_{n,j})}(K_{n,j}) &= d_{n+1}^{\frac{2}{q-1} - N} R_{2/q,q'}^{B_{2\delta_n d_{n+1}}(d_{n+1} a_{n,j})}(\tilde{K}_{n,j}) \\ &\approx d_{n+1}^{\frac{2}{q-1} - N} C_{2/q,q'}^{B_{2\delta_n d_{n+1}}(d_{n+1} a_{n,j})}(\tilde{K}_{n,j}) \\ &\approx d_{n+1}^{\frac{2}{q-1} - N} C_{2/q,q'}(\tilde{K}_{n,j}) \end{aligned} \tag{2.80}$$

where $\tilde{K}_{n,j} = d_{n+1} K_{n,j}$. For a fixed $n > 0$ and each repartition Λ of points $\tilde{a}_{n,j} = d_{n+1}\, a_{n,j}$ such that the balls $B_{2^\theta}(\tilde{a}_{n,j})$ are disjoint, the quasi-additivity property holds: if we set

$$K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} K_{n,j}\ , \quad \tilde{K}_{n,\Lambda} = d_{n+1}\, K_{n,\Lambda} = \bigcup_{a_{n,j} \in \Lambda} \tilde{K}_{n,j} \quad \text{and} \quad \tilde{K}_n = d_{n+1}\, K_n,$$

then

$$\sum_{a_{n,j}\in\Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \approx C_{2/q,q'}(\tilde{K}_{n,\Lambda}). \tag{2.81}$$

The maximal cardinal of any such repartition Λ is of the order of Cn^{N-1} for some positive constant $C = C(N)$, therefore, the number of repartitions needed for a full covering of the set $\tilde{T}_n$ is of finite order depending upon the dimension. Because $\tilde{K}_n$ is the union of the $\tilde{K}_{n,\Lambda}$,

$$\sum_{a_{n,j}\in\Theta_n} C_{2/q,q'}(\tilde{K}_{n,j}) = \sum_{\Lambda}\sum_{a_{n,j}\in\Lambda} C_{2/q,q'}(\tilde{K}_{n,j}) \approx C_{2/q,q'}(\tilde{K}_n). \tag{2.82}$$

By Lemma 2.9,

$$C_{2/q,q'}(\tilde{K}_n) \leq C^{B_{2d_{n+1}}}_{2/q,q'}(\tilde{K}_n) \approx d_{n+1}^{N-\frac{1}{q-1}} C^{B_2}_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right) \approx d_{n+1}^{N-\frac{1}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right),$$

we obtain (2.78) by combining this last inequality with (2.80) and (2.82). □

Proof of Theorem 2.22. Step 1. We first notice that

$$\overline{u}_K \leq \sum_{0\leq n\leq a_K}\sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j}}. \tag{2.83}$$

Actually, since $K = \bigcup_n \bigcup_{a_{n,j}} K_{n,j}$, for any $0 < \epsilon' < \epsilon$, there holds $\overline{K_{\epsilon'}} \subset \bigcup_n \bigcup_{a_{n,j}} K_{n,j\,\epsilon}$. Because a finite sum of positive solutions of (1.1) is a super solution,

$$\overline{u}_{K_{\epsilon'}} \leq \sum_{0\leq n\leq a_K}\sum_{a_{n,j}\in\Theta_n} \overline{u}_{K_{n,j\,\epsilon}}. \tag{2.84}$$

Letting successively ϵ' and ϵ go to 0 implies (2.83).

Step 2. Let $n \in \mathbb{N}$. Since $K_{n,j} \subset B_{\delta_n}(a_{n,j})$ and $|x - a_{n,j}| = (d_n + d_{n+1})/2$, we can apply the previous lemmas with $r = \delta_n$ and $\rho = r$. For $n \geq n_N$, there holds $t = 1 \geq (r + 2\rho)^2 = 9/(n+1)$ and $|x - a_{n,j}| = (\sqrt{n+1} - \sqrt{n})/2 \geq (2 + C_N)(3/\sqrt{n+1})$ (notice that $n_N \geq 8$). Thus

$$u_{K_{n,j}}(0,1) \leq Ce^{\left(\sqrt{n}-3/\sqrt{n+1}\right)^2/4} R^{B_{2\delta_n}(a_{n,j})}_{2/q,q'}(K_{n,j}) \leq Ce^{3/2}e^{-\frac{n}{4}} R^{B_{2\delta_n}(a_{n,j})}_{2/q,q'}(K_{n,j}). \tag{2.85}$$

Using Lemma 2.23 we obtain, with $d_n = d_n(1) = \sqrt{n+1}$

$$\sum_{n=n_N}^{a_K}\sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) \leq C\sum_{n=n_N}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{2.86}$$

Finally, we apply Lemma 2.14 if $1 \leq n < n_N$ and get

$$\begin{aligned}\sum_{1}^{n_N-1}\sum_{a_{n,j}\in\Theta_n} u_{K_{n,j}}(0,1) &\leq C\sum_{1}^{n_N-1} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right)\\ &\leq C'\sum_{1}^{n_N-1} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right).\end{aligned} \tag{2.87}$$

For $n = 0$, we proceed similarly, in splitting K_1 in a finite number of $K_{1,i}$, depending only on the dimension, such that $\operatorname{diam} K_{1,i} < 1/3$. Combining (2.86) and (2.87), we derive

$$\overline{u}_K(0,1) \leq C \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{2.88}$$

In order to derive the same result for any $t > 0$, we notice that

$$\overline{u}_K(y,t) = t^{-\frac{1}{q-1}} \overline{u}_{K/\sqrt{t}}(y/\sqrt{t}, 1).$$

Going back to the definition of $d_n = d_n(K,t) = \sqrt{nt} = d_n(K\sqrt{t}, 1)$, we derive from (2.88) and the fact that $a_{K,t} = a_{K\sqrt{t},1}$

$$\overline{u}_K(0,t) \leq C t^{-\frac{N}{2}} \sum_{n=0}^{a_K} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{2.89}$$

with $d_n = d_n(t) = \sqrt{t(n+1)}$. This is (2.77) with $x = 0$, and a space translation leads to the final result. $\square$

Proof of Theorem 2.1. Let $m > 0$ and $F_m = F \cap \overline{B}_m$. We denote by $U_{B^c_m}$ the maximal solution of (1.1) in Q_∞ the initial trace of which vanishes on B_m. Such a solution is actually the unique solution of (2.1) which satisfies

$$\lim_{t\to 0} u(x,t) = \infty$$

uniformly on $B^c_{m'}$, for any $m' > m$: this can be easily proved by noticing that

$$U_{B^c_m\,\ell}(y,t) = \ell^{\frac{1}{q-1}} U_{B^c_m}(\sqrt{\ell} y, \ell t) = U_{B^c_{m/\sqrt{\ell}}}(y,t).$$

Furthermore

$$\lim_{m\to\infty} U_{B^c_m}(y,t) = \lim_{m\to\infty} m^{-\frac{2}{q-1}} U_{B^c_1}(y/m, t/m^2) = 0$$

uniformly on any compact subset of $\overline{Q}_\infty$. Since $\overline{u}_{F_m} + U_{B^c_m}$ is a super-solution, it is larger that $\overline{u}_F$ and therefore $\overline{u}_{F_m} \uparrow \overline{u}_F$. Because $W_{F_m}(x,t) \leq W_F(x,t)$ and $\overline{u}_{F_m} \leq C_1 W_{F_m}(x,t)$, the result follows. $\square$

Remark. It is clear that Theorem 2.1 still holds if u is a positive subsolution of (1.1) satisfying the initial trace condition (1.21).

Theorem 2.1 admits the following integral expression.

Theorem 2.24 *Assume $q \geq q_c$. Then there exists a positive constant $C_1^* = C^*(N,q,T)$ such that, for any closed subset F of $\mathbb{R}^N$, there holds*

$$\overline{u}_F(x,t) \leq \frac{C_1^*}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{1}{s} F \cap B_1(x)\right) s\, ds, \tag{2.90}$$

where $a_t = \min\{n : F \subset B_{\sqrt{n+1)t}}(x)\}$.

Proof. We first use

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \le C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right),$$

and we denote

$$\Phi(s) = C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \qquad \forall s > 0. \tag{2.91}$$

Step 1. The following inequality holds

$$c_1\Phi(\alpha s) \le \Phi(s) \le c_2\Phi(\beta s) \qquad \forall s > 0, \quad \forall 1/2 \le \alpha \le 1 \le \beta \le 2, \tag{2.92}$$

for some positive constants c_1, c_2 depending on N and q. See [1] and [32]. If $\beta \in [1, 2]$,

$$\Phi(\beta s) = C_{2/q,q'}\left(\frac{1}{\beta}\left(\frac{F}{s} \cap B_\beta\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\beta\right) \ge c_1\Phi(s).$$

If $\alpha \in [1/2, 1]$,

$$\Phi(\alpha s) = C_{2/q,q'}\left(\frac{1}{\alpha}\left(\frac{F}{s} \cap B_\alpha\right)\right) \approx C_{2/q,q'}\left(\frac{F}{s} \cap B_\alpha\right) \le c_2\Phi(s).$$

Step 2. By (2.92)

$$C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \le c_2 C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) \qquad \forall\, s \in [d_{n+1}, d_{n+2}],$$

and $n \le a_t$. Then

$$\begin{aligned} c_2\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1\right) s\,ds \\ \ge C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) \int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-s^2/4t} s\,ds. \end{aligned}$$

Using the fact that $N - \frac{2}{q-1} \ge 0$, we get,

$$\int_{d_{n+1}}^{d_{n+2}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} s\,ds \ge e^{-\frac{n+2}{4}} d_{n+1}^{N-\frac{2}{q-1}+1}(d_{n+2} - d_{n+1}) \tag{2.93}$$

$$\ge \frac{t}{4e^2} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}}. \tag{2.94}$$

Thus

$$\overline{u}_F(x,t) \le \frac{C}{t^{1+\frac{N}{2}}} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{1}{s}F \cap B_1\right) s\,ds, \tag{2.95}$$

which ends the proof. $\square$

3 Estimate from below

If $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N)$, we denote by $u_\mu = u_{\mu,0}$ the solution of

$$\left\{\begin{array}{lll} \partial_t u_\mu - \Delta u_\mu + u^q_\mu = 0 & & \text{in } Q_T, \\ u_\mu(.,0) = \mu & & \text{in } \mathbb{R}^N. \end{array}\right. \tag{3.1}$$

The maximal σ-moderate solution of (1.1) which has an initial trace vanishing outside a closed set F is defined by

$$\underline{u}_F = \sup\left\{u_\mu : \mu \in \mathfrak{M}^q_+(\mathbb{R}^N) \cap \mathfrak{M}^b(\mathbb{R}^N)\,,\ \mu(F^c) = 0\right\}. \tag{3.2}$$

The main result of this section is the next one

Theorem 3.1 *Assume $q \geq q_c$. There exists a constant $C_2 = C_2(N,q,T) > 0$ such that, for any closed subset $F \subset \mathbb{R}^N$, there holds*

$$\underline{u}_F(x,t) \geq C_2 W_F(x,t) \qquad \forall (x,t) \in Q_T. \tag{3.3}$$

We first assume that F is compact, and we denote it by K. The first observation is that if $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N)$, $u_\mu \in L^q(Q_T)$ (see lemma below) and $0 \leq u_\mu \leq \mathbb{H}[\mu] := \mathbb{H}_\mu$. Therefore

$$u_\mu \geq \mathbb{H}[\mu] - \mathbb{G}\left[\mathbb{H}[\mu]^q\right], \tag{3.4}$$

where $\mathbb{G}$ is the parabolic Green potential in Q_T defined by

$$\mathbb{G}[f](t) = \int_0^t \mathbb{H}[f(s)](t-s)ds = \int_0^t \int_{\mathbb{R}^N} H(.,y,t-s)f(y,s)dyds.$$

The main idea of the proof is as follows. For any $(x,t) \in Q_T$, construct a measure $\mu = \mu(x,t) \in \mathfrak{M}^q_+(\mathbb{R}^N)$ such that there holds

$$\mathbb{H}[\mu](x,t) \geq C W_K(x,t) \qquad \forall (x,t) \in Q_T, \tag{3.5}$$

and

$$\mathbb{G}\left(\mathbb{H}[\mu]\right)^q \leq C\,\mathbb{H}[\mu] \quad \text{in } Q_T, \tag{3.6}$$

with constants C depends only on N, q, and T. Then replace μ by $\mu_\epsilon = \epsilon\mu$ with $\epsilon = (2C)^{-\frac{1}{q-1}}$ in order to derive

$$u_{\mu_\epsilon} \geq 2^{-1}\mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \tag{3.7}$$

From this follows

$$\underline{u}_K \geq 2^{-1}\mathbb{H}_{\mu_\epsilon} \geq 2^{-1} C W_K. \tag{3.8}$$

and the proof of Theorem 3.1 with $C_2 = 2^{-1}C$. In the following sections we describe the construction of measures $\mu(x,t)$ satisfying (3.5) and (3.6).

3.1 Estimate from below of the solution of the heat equation

The purely spatial slicing used is the trace on $\mathbb{R}^N \times \{0\}$ of an *extended slicing* in Q_T which is constructed as follows: if K is a compact subset of $\mathbb{R}^N$, $m = (x,t)$, we define d_K, λ, d_n and a_t as in Section 2.3. Let $\alpha \in (0,1)$ to be fixed later on, we define $\mathcal{T}_n$ for $n \in \mathbb{Z}$ by

$$\mathcal{T}_n = \begin{cases} \mathcal{B}^2_{\sqrt{t(n+1)}}(m) \setminus \mathcal{B}^2_{\sqrt{tn}}(m) & \text{if } n \geq 1, \\ \mathcal{B}^2_{\alpha^{-n}\sqrt{t}}(m) \setminus \mathcal{B}^2_{\alpha^{1-n}\sqrt{t}}(m) & \text{if } n \leq 0, \end{cases}$$

and put

$$\mathcal{T}^*_n = \mathcal{T}_n \cap \{s : 0 \leq s \leq t\}, \quad \text{for } n \in \mathbb{Z}.$$

We recall that for $n \in \mathbb{N}_*$,

$$\mathcal{Q}_n = K \cap \mathcal{B}^2_{\sqrt{t(n+1)}}(m) = K \cap B_{d_n}(x)$$

and

$$K_n = K \cap \mathcal{T}_{n+1} = K \cap \left(B_{d_{n+1}}(x) \setminus B_{d_n}(x)\right).$$

Let $\nu_n \in \mathfrak{M}^b_+(\mathbb{R}^N) \cap W^{-2/q,q}(\mathbb{R}^N)$ be the *q-capacitary measure* of the set K_n/d_{n+1}. See [1, Sec. 2.2]. Such a measure has support in K_n/d_{n+1} and

$$\nu_n(K_n/d_{n+1}) = C_{2/q,q'}(K_n/d_{n+1}) \text{ and } \|\nu_n\|_{W^{-2/q,q'}(\mathbb{R}^N)} = \left(C_{2/q,q'}(K_n/d_{n+1})\right)^{1/q}. \tag{3.9}$$

We define μ_n as follows

$$\mu_n(A) = d_{n+1}^{N-\frac{2}{q-1}} \nu_n(A/d_{n+1}) \qquad \forall A \subset K_n,\ A \text{ Borel}, \tag{3.10}$$

and set

$$\mu_{t,K} = \sum_{n=0}^{a_t} \mu_n,$$

and

$$\mathbb{H}_{\mu_{t,K}} = \sum_{n=0}^{a_t} \mathbb{H}_{\mu_n}. \tag{3.11}$$

Proposition 3.2 *Let $q \geq q_c$, then there holds*

$$\mathbb{H}_{\mu_{t,K}}(x,t) \geq \frac{1}{(4\pi t)^{\frac{N}{2}}} \sum_{n=0}^{a_t} e^{-\frac{n+1}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{3.12}$$

in $\mathbb{R}^N \times (0,T)$.

Proof. Since

$$\mathbb{H}_{\mu_n}(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{K_n} e^{-\frac{|x-y|^2}{4t}} d\mu_n, \tag{3.13}$$

and

$$y \in K_n \Longrightarrow |x-y| \leq d_{n+1},$$

(3.12) follows because of (3.10) and (3.11). $\square$

3.2 Estimate from above of the nonlinear term

We write (3.4) under the form

$$\begin{aligned} u_\mu(x,t) &\geq \sum_{n\in\mathbb{Z}} \mathbb{H}_{\mu_n}(x,t) - \int_0^t\int_{\mathbb{R}^N} H(x,y,t-s)\left[\sum_{n\in A_K}\mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\ &= I_1 - I_2. \end{aligned} \tag{3.14}$$

since $\mu_n = 0$ if $n\notin A_K = \mathbb{N}\cap[1,a_t]$, and

$$\begin{aligned} I_2 &= \frac{1}{(4\pi)^{\frac{N}{2}}}\int_0^t\int_{\mathbb{R}^N}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\sum_{n\in A_K}\mathbb{H}_{\mu_n}(y,s)\right]^q dyds \\ &= \frac{1}{(4\pi)^{\frac{N}{2}}}(J_\ell + J'_\ell), \end{aligned} \tag{3.15}$$

for some $\ell\in\mathbb{N}^*$ to be fixed later on, where

$$J_\ell = \sum_{p\in\mathbb{Z}}\iint_{\mathcal{T}_p^*}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\sum_{n<p+\ell}\mathbb{H}_{\mu_n}(y,s)\right]^q dyds,$$

and

$$J'_\ell = \sum_{p\in\mathbb{Z}}\iint_{\mathcal{T}_p^*}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\sum_{n\geq p+\ell}\mathbb{H}_{\mu_n}(y,s)\right]^q dyds.$$

The next estimate will be used several times in the sequel.

Lemma 3.3 *Let $0<a<b$ and $t>0$, then,*

$$\max\left\{\sigma^{-\frac{N}{2}}e^{-\frac{\rho^2}{4\sigma}} : 0\leq\sigma\leq t,\ at\leq\rho^2+\sigma\leq bt\right\} = e^{\frac{1}{4}}\begin{cases} t^{-\frac{N}{2}}e^{-\frac{a}{4}} & \text{if } \dfrac{a}{2N}>1, \\ \left(\dfrac{2N}{at}\right)^{\frac{N}{2}}e^{-\frac{N}{2}} & \text{if } \dfrac{a}{2N}\leq 1. \end{cases}$$

Proof. Set

$$\mathcal{J}(\rho,\sigma) = \sigma^{-\frac{N}{2}}e^{-\frac{\rho^2}{4\sigma}}$$

and

$$\mathcal{K}_{a,b,t} = \left\{(\rho,\sigma)\in[0,\infty)\times(0,t]:\ at\leq\rho^2+\sigma\leq bt\right\}.$$

We first notice that, for fixed σ, the maximum of $\mathcal{J}(.,\sigma)$ is achieved for ρ minimal. If $\sigma\in[at,bt]$ the minimal value of ρ is 0, while if $\sigma\in(0,at)$, the minimum of ρ is $\sqrt{at-s}$.

- Assume first $a\geq 1$, then $\mathcal{J}(\sqrt{at-\sigma},\sigma) = e^{\frac{1}{4}}\sigma^{-\frac{N}{4}}e^{-\frac{at}{4\sigma}}$. Thus if $1\leq a/2N$, the minimal value of $\mathcal{J}(\sqrt{at-\sigma},\sigma)$ is $e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}$, while if $a/2N<1\leq a$, the minimum is $e^{\frac{1}{4}}t^{-\frac{N}{2}}e^{-\frac{a}{4}}$.

- Assume now $a \leq 1$. Then

$$\max\{\mathcal{J}(\rho,\sigma):\ (\rho,\sigma)\in\mathcal{K}_{a,b,t}\} = \max\left\{\max_{\sigma\in(at,t]}\mathcal{J}(0,\sigma),\ \max_{\sigma\in(0,at]}\mathcal{J}(\sqrt{at-\sigma},\sigma)\right\}$$
$$= \max\left\{(at)^{-\frac{N}{2}},\ e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}\right\}$$
$$= e^{\frac{1-2N}{4}}\left(\frac{2N}{at}\right)^{\frac{N}{2}}.$$

Combining these two estimates, we derive the result. □

Remark. The following variant of Lemma 3.3 will be useful in the sequel: *For any $\theta \geq 1/2N$ there holds*

$$\max\{\mathcal{J}(\rho,\sigma):\ (\rho,\sigma)\in\mathcal{K}(a,b,t)\} \leq e^{\frac{1}{4}}\left(\frac{2N\theta}{t}\right)^{\frac{N}{2}} e^{-\frac{a}{4}} \quad \text{if } \theta a \geq 1. \tag{3.16}$$

Lemma 3.4 *There exists a positive constant $C = C(N,\ell,q)$ such that*

$$J_\ell \leq Ct^{-\frac{N}{2}}\sum_{n=1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-(1+(n-\ell)_+)/4}\, C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.17}$$

Proof. The set of the p's for the summation in J_ℓ is reduced to $\mathbb{Z}\cap[-\ell+2,\infty)$, thus we write

$$J_\ell = J_{1,\ell} + J_{2,\ell}$$

where

$$J_{1,\ell} = \sum_{p=2-\ell}^{0}\iint_{\mathcal{T}_p^*}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\sum_{n<p+\ell}\mathbb{H}_{\mu_n}(y,s)\right]^q$$

and

$$J_{2,\ell} = \sum_{p=1}^{\infty}\iint_{\mathcal{T}_p^*}(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}\left[\sum_{n<p+\ell}\mathbb{H}_{\mu_n}(y,s)\right]^q.$$

If $p = 2-\ell,\ldots,0$,

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow t\alpha^{2-2p} \leq |x-y|^2+t-s \leq t\alpha^{-2p},$$

and, if $p \geq 1$

$$(y,s)\in\mathcal{T}_p^* \Longrightarrow pt \leq |x-y|^2+t-s \leq (p+1)t.$$

By Lemma 3.3 and (3.16), there exists $C = C(N,\ell,\alpha) > 0$ such that

$$\max\left\{(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}:(y,s)\in\mathcal{T}_p^*\right\} \leq Ct^{-\frac{N}{2}}e^{-\alpha^{2-2p}/4}, \tag{3.18}$$

if $p = 2-\ell,\ldots,0$, and

$$\max\left\{(t-s)^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4(t-s)}}:(y,s)\in\mathcal{T}_p^*\right\} \leq Ct^{-\frac{N}{2}}e^{-p/4}, \tag{3.19}$$

if $p \geq 1$. When $p = 2 - \ell, \ldots, 0$

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \leq C \sum_{1}^{p+\ell-1} \mathbb{H}^q_{\mu_n}(y,s), \tag{3.20}$$

for some $C = C(\ell, q) > 0$, thus

$$\begin{aligned} J_{1,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=2-\ell}^{0} e^{-\frac{\alpha^{2-2p}}{4}} \sum_{n=1}^{p+\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \sum_{p=n-\ell+1}^{0} e^{-\frac{\alpha^{2-2p}}{4}} \\ &\leq Ct^{-\frac{N}{2}} e^{-\frac{\alpha^{2\ell-2}}{4}} \sum_{n=1}^{\ell-1} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)}. \end{aligned} \tag{3.21}$$

If the set of p's is not upper bounded, we introduce some parameter $\delta > 0$ to be made precise later on. Then

$$\left[\sum_{1}^{p+\ell-1} \mathbb{H}_{\mu_n}(y,s)\right]^q \leq \left[\sum_{1}^{p+\ell-1} e^{\delta q' \frac{n}{4}}\right]^{q/q'} \sum_{1}^{p+\ell-1} e^{-\frac{\delta q n}{4}} \mathbb{H}^q_{\mu_n}(y,s), \tag{3.22}$$

with $q' = q/(q-1)$. If, by convention $\mu_n = 0$ whenever $n > a_t$, we obtain, for some $C > 0$ which depends also on δ,

$$\begin{aligned} J_{2,\ell} &\leq Ct^{-\frac{N}{2}} \sum_{p=1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \sum_{n=1}^{p+\ell-1} e^{-\frac{\delta q n}{4}} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)} e^{-\frac{\delta q n}{4}} \sum_{p=(n-\ell+1)\vee 1}^{\infty} e^{\frac{\delta(p+\ell-1)q-p}{4}} \\ &\leq Ct^{-\frac{N}{2}} \sum_{n=1}^{\infty} e^{-\frac{1+(n-\ell)_+}{4}} \|\mathbb{H}_{\mu_n}\|^q_{L^q(Q_t)}. \end{aligned} \tag{3.23}$$

Notice that we choose δ such that $\delta \ell q < 1$. Combining (3.21) and (3.23), we derive (3.17) from Lemma 2.11, (3.9) and (3.10). $\square$

The set of indices p for which the μ_n terms are not zero in J'_ℓ is $\mathbb{Z} \cap (-\infty, a_t - \ell]$. We write

$$J'_\ell = J'_{1,\ell} + J'_{2,\ell},$$

where

$$J'_{1,\ell} = \sum_{p=-\infty}^{0} \iint_{\mathcal{T}^*_p} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=1\vee p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s)\right]^q dyds,$$

and

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \mathbb{H}_{\mu_n}(y,s) \right]^q dyds.$$

Lemma 3.5 *There exists a constant $C = C(N,q,\ell) > 0$ such that*

$$J'_{1,\ell} \leq Ct^{1-\frac{Nq}{2}} \sum_{n=0}^{a_t} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} d_{n+1}^{Nq-2q'} C^q_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right), \tag{3.24}$$

where $\beta_0 = (q-1)/4$ and $h = 2q(q+1)/(q-1)^2$.

Proof. Since

$$(y,s) \in \mathcal{T}_p^*, \text{ and } (z,0) \in K_n \Longrightarrow |y-z| \geq (\sqrt{n} - \alpha^{-p})\sqrt{t}, \tag{3.25}$$

there holds

$$\mathbb{H}_{\mu_n}(y,s) \leq (4\pi s)^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2 t}{4s}} \mu_n(K_n) \leq Ct^{-\frac{N}{2}} e^{-\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n(K_n),$$

by Lemma 3.3. Let $\{\epsilon_n\}$ be a sequence of positive numbers such that

$$A_\epsilon = \sum_{n=1}^{\infty} \epsilon_n^{q'} < \infty,$$

then

$$\begin{aligned}
J'_{1,\ell} &\leq CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{p=-\infty}^{0} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \sum_{n=1\vee(p+\ell)}^{\infty} \epsilon_n^{-q} e^{-q\frac{(\sqrt{n}-\alpha^{-p})^2}{4}} \mu_n^q(K_n) ds\, dy \\
&\leq CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) \sum_{-\infty}^{p=0\wedge(n-\ell)} e^{-\frac{q(\sqrt{n}-\alpha^{-p})^2}{4}} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds\, dy \\
&\leq CA_\epsilon^{q/q'} t^{-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-\frac{q(\sqrt{n}-1)^2}{4}} \iint_{\{\cup_{p\leq 0}\mathcal{T}_p^*\}} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} ds\, dy \\
&\leq CA_\epsilon^{q/q'} t^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} \epsilon_n^{-q} \mu_n^q(K_n) e^{-\frac{q(\sqrt{n}-1)^2}{4}}.
\end{aligned} \tag{3.26}$$

Set $h = 2q(q+1)/(q-1)^2$ and $Q = (1+q)/2$, then $q(\sqrt{n}-1)^2 \geq Q(n-h)_+$ for any $n \geq 1$. If we choose $\epsilon_n = e^{-\frac{(q-1)(n-h)_+}{16q}}$, there holds $\epsilon_n^{-q} e^{-\frac{q(\sqrt{n}-1)^2}{4}} \leq e^{-\frac{(q+3)(n-h)_+}{16}}$. Finally

$$J'_{1,\ell} \leq Ct^{1-\frac{Nq}{2}} \sum_{n=1}^{\infty} e^{-\frac{(1+\beta_0)(n-h)_+}{4}} \mu_n^q(K_n),$$

with $\beta_0 = (q-1)/4$, which yields to (3.24) by the choice of the μ_n. □

In order to make easier the obtention of the estimate of the term $J'_{2,\ell}$, we first give the proof in dimension 1.

Lemma 3.6 *Assume $N = 1$ and ℓ is an integer larger than 1. There exists a positive constant $C = C(q, \ell) > 0$ such that*

$$J'_{2,\ell} \le Ct^{-1/2} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'} \left(\frac{K_n}{d_{n+1}} \right). \tag{3.27}$$

Proof. If $(y, s) \in \mathcal{T}_p^*$ and $z \in K_n$ $(p \ge 1,\ n \ge p = \ell)$, there holds $|x - y| \ge \sqrt{t}\sqrt{p}$ and $|y - z| \ge \sqrt{t}(\sqrt{n} - \sqrt{p+1})$. Therefore

$$J'_{2,\ell} \le C\sqrt{t} \sum_{p=1}^{a_t - \ell} \frac{1}{\sqrt{p}} \int_0^t e^{-\frac{pt}{4(t-s)}} \left(\sum_{n=p+\ell}^{a_t} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q.$$

If $\epsilon \in (0, q)$ is some positive parameter which will be made more precise later on, there holds

$$\begin{aligned}
&\left(\sum_{n=p+\ell}^{a_t} s^{-1/2} e^{-\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n(K_n) \right)^q \\
&\qquad \le \left(\sum_{n=p+\ell}^{a_t} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \right)^{q/q'} \sum_{n=p+\ell}^{a_t} s^{-\frac{q}{2}} e^{-(q-\epsilon) \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \mu_n^q(K_n),
\end{aligned}$$

by Hölder's inequality. By comparison between series and integrals and using Gauss integral

$$\begin{aligned}
\sum_{n=p+\ell}^{a_t} e^{-\epsilon q' \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} &\le \int_{p+\ell}^{\infty} e^{-\epsilon q' \frac{(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\
&= 2 \int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} (x + \sqrt{p+1}) dx \\
&\le \frac{4s}{\epsilon q' t} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{4s}} + 2\sqrt{p+1} \int_{\sqrt{p+\ell}-\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q' x^2 t}{4s}} dx \\
&\le C \sqrt{\frac{(p+1)s}{t}} e^{-\epsilon q' \frac{(\sqrt{p+\ell}-\sqrt{p+1})^2 t}{2s}} \\
&\le C \sqrt{\frac{(p+1)s}{t}}.
\end{aligned}$$

If we set $q_\epsilon = q - \epsilon$, then

$$J'_{2,\ell} \le C\epsilon^{-q'/q} t^{1-\frac{q}{2}} \sum_{n=\ell+1}^{\infty} \mu_n^q(K_n) \sum_{p=1}^{n-\ell} p^{\frac{q-2}{2}} \int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds.$$

where $C = C(\epsilon, q) > 0$. Since

$$\begin{aligned}
&\int_0^t (t-s)^{-1/2} s^{-1/2} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \\
&\qquad = \int_0^1 (1-s)^{-1/2} s^{-1/2} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}} ds,
\end{aligned}$$

we can apply Lemma A.1 with $a = 1/2$, $b = 1/2$, $A = \sqrt{p}$ and $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$. In this range of indices $B \geq \sqrt{q_\epsilon}(\sqrt{p+\ell} - \sqrt{p+1}) \geq \frac{\sqrt{q_\epsilon}(\ell-1)}{\sqrt{p}}$, thus $\kappa = \sqrt{q_\epsilon}(\ell - 1)$ and

$$\sqrt{\frac{A}{A+B}}\sqrt{\frac{B}{A+B}} \leq p^{\frac{1}{4}} n^{-1/2}(\sqrt{n} - \sqrt{p})^{1/2}.$$

Therefore

$$\int_0^t (t-s)^{-1/2} s^{-\frac{q}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q\frac{(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} ds \leq \frac{Cp^{\frac{1}{4}}(\sqrt{n} - \sqrt{p})^{1/2}}{\sqrt{n}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \tag{3.28}$$

which implies

$$J'_{2,\ell} \leq Ct^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} \frac{\mu_n^q(K_n)}{\sqrt{n}} \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n} - \sqrt{p})^{1/2} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}, \tag{3.29}$$

where C depends of ϵ, q and ℓ. By Lemma A.2

$$J'_{2,\ell} \leq Ct^{1-\frac{q}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{q-3}{2}} e^{-\frac{n}{4}} \mu_n^q(K_n) \tag{3.30}$$

Because $\mu_n(K_n) = d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right)$ (remember $N = 1$) and diam $\frac{K_n}{d_{n+1}} \leq n^{-1}$, there holds

$$\mu_n^q(K_n) \leq C\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{q-3} \mu_n(K_n) = C\left(\frac{\sqrt{t}}{\sqrt{n}}\right)^{q-3} d_{n+1}^{\frac{q-3}{q-1}} C_{2/q,q'}(K_n/d_{n+1}) \tag{3.31}$$

and inequality (3.27) follows. $\square$

Next we give the general proof. For this task we will use again the quasi-additivity with separated partitions.

Lemma 3.7 *Assume $N \geq 2$ and ℓ is an integer larger than 1. There exists a positive constant $C_1 = C_1(q, N, \ell) > 0$ such that*

$$J'_{2,\ell} \leq C_1 t^{-\frac{N}{2}} \sum_{n=\ell}^{a_t} e^{-\frac{n}{4}} d_{n+1}^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right). \tag{3.32}$$

Proof. As in the proof of Theorem 2.22, we know that there exists a finite number J, depending only on the dimension N, of separated sub-partitions $\{\#\Theta_{t,n}^h\}_{h=1}^J$ of the rescaled sets $\tilde{T}_n = \sqrt{\frac{n+1}{t}} T_n$ by the N-dim balls $B_2(\tilde{a}_{n,j})$ where $\tilde{a}_{n,j} = \sqrt{\frac{n+1}{t}} a_{n,j}$, $|a_{n,j}| = \dfrac{d_{n+1} + d_n}{2}$ and $|a_{n,j} - a_{n,k}| \geq \sqrt{\frac{4t}{n+1}}$. Furthermore $\#\Theta_{t,n}^h \leq Cn^{N-1}$. We denote $K_{n,j} = K_n \cap B_{\sqrt{\frac{t}{n+1}}}(a_{n,j})$.

We write $\mu_n = \sum_{h=1}^{J} \mu_n^h$, and accordingly $J'_{2,\ell} = \sum_{h=1}^{J} {J'}_{2,\ell}^{h}$, where $\mu_n^h = \sum_{j\in\Theta_{t,n}^h} \mu_{n,j}$, and $\mu_{n,j}$ are the capacitary measures of $K_{n,j}$ relative to $B_{n,j} = B_{6t/5\sqrt{n}}(a_n, j)$, which means

$$\nu_{n,j}(K_{n,j}) = C_{2/q,q'}^{B_{n,j}}(K_{n,j}) \quad \text{and} \quad \|\nu_{n,j}\|_{W^{-2/q,q'}(B_{n,j})} = \left(C_{2/q,q'}^{B_{n,j}}(K_{n,j})\right)^{1/q}. \tag{3.33}$$

Thus

$$J'_{2,\ell} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{h=1}^{J} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s)\right]^q dyds.$$

We denote

$${J'}_{2,\ell}^{h} = \sum_{p=1}^{a_t-\ell} \iint_{\mathcal{T}_p^*} (t-s)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\sum_{n=p+\ell}^{\infty} \sum_{j\in\Theta_{t,n}^h} \mathbb{H}_{\mu_{n,j}}(y,s)\right]^q dyds,$$

and clearly

$$J'_{2,\ell} \leq C \sum_{h=1}^{J} {J'}_{2,\ell}^{h}, \tag{3.34}$$

where C depends only on N and q. For integers n and p such that $n \geq \ell + 1$, we set

$$\lambda_{n,j,y} = \inf\{|y-z| : z \in B_{\sqrt{t}/\sqrt{n+1}}(a_{n,j})\} = |y - a_{n,j}| - \frac{\sqrt{t}}{\sqrt{n+1}}.$$

Therefore

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} \int_{K_n} e^{-\frac{|y-z|^2}{4s}} d\mu_n^h(z) &= \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} \int_{K_{n,j}} e^{-\frac{|y-z|^2}{4s}} d\mu_{n,j}(z) \\ &\leq \left(\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}}\right)^{1/q'} \left(\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} e^{-q\lambda_{n,j,y}^2 \frac{1-\epsilon}{4s}} \mu_{n,j}^q(K_{n,j})\right)^{1/q}\end{aligned}$$

where $\epsilon > 0$ will be made precise later on.

Step 1 We claim that

$$\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\epsilon q' \frac{\lambda_{n,j,y}^2}{4s}} \leq C\sqrt{\frac{ps}{t}} \tag{3.35}$$

where C depends on ϵ, q and N. If y is fixed in T_p, we denote by z_y the point of T_n which solves $|y - z_y| = \mathrm{dist}\,(y, T_n)$. Thus

$$\sqrt{t}(\sqrt{n} - \sqrt{p+1}) \leq |y - z_y| \leq t(\sqrt{n} - \sqrt{p}).$$

Let $Y = y\sqrt{t(p+1)}/|y|$. On the axis $\overrightarrow{0Y}$ we set $\mathbf{e} = Y/|Y|$, consider the points $b_k = (k\sqrt{t}/\sqrt{n})\mathbf{e}$ where $-n \le k \le n$ and denote by $G_{n,k}$ the spherical shell obtained by intersecting the spherical shell T_n with the domain $H_{n,k}$ which is the set of points in $\mathbb{R}^N$ limited by the hyperplanes orthogonal to $\overrightarrow{0Y}$ going through $((k+1)\sqrt{t}/\sqrt{n})\mathbf{e}$ and $((k-1)\sqrt{t}/\sqrt{n})\mathbf{e}$. The number of points $a_{n,j} \in G_{n,k}$ is smaller than $C(n+1-|k|)^{N-2}$, where C depends only on N, and we denote by $\Lambda_{n,k}$ the set of $j \in \Theta_{t,n}$ such that $a_{n,j} \in G_{n,k}$. Furthermore, if $a_{n,j} \in G_{n,k}$ elementary geometric considerations (Pythagora's theorem) imply that $\lambda^2_{n,j,y}$ is greater than $t(n+p+1-2k\sqrt{p+1}/\sqrt{n})$. Therefore

$$\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\epsilon q' \frac{\lambda^2_{n,j,y}}{4s}} \le C \sum_{n=p+\ell}^{a_t} \sum_{k=-n}^{n} (n+1-|k|)^{N-2} e^{-\frac{\epsilon q'(n+p+1-2k\sqrt{p+1}/)t}{4s\sqrt{n}}}. \tag{3.36}$$

Case $N=2$. Summing a geometric series and using the inequality $\frac{e^u}{e^u-1} \le 1+u^{-1}$ for $u>0$, we obtain

$$\begin{aligned}\sum_{k=-n}^{n} e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}} &\le e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \frac{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}}{e^{\frac{\epsilon q' t\sqrt{p+1}}{2s\sqrt{n}}}-1} \\ &\le e^{\frac{\epsilon q' t\sqrt{n(p+1)}}{2s}} \left(1+\frac{2s\sqrt{n}}{\epsilon q' t\sqrt{p+1}}\right).\end{aligned} \tag{3.37}$$

Thus, by comparison between series and integrals,

$$\begin{aligned}\sum_{n=p+\ell}^{a_t} \sum_{j \in \Theta_{t,n}} e^{-\frac{\epsilon q' \lambda^2_{n,j,y}}{4s}} &\le C \sum_{n=p+\ell}^{a_t} \left(1+\frac{s\sqrt{n}}{t\sqrt{p}}\right) e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2}{4s}} \\ &\le C \int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\ &\qquad + \frac{Cs}{t\sqrt{p}} \int_{p+1}^{\infty} \sqrt{x} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx.\end{aligned} \tag{3.38}$$

Next

$$\begin{aligned}\int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}} y\,dy \\ &= 2\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} y\,dy + 2\sqrt{p+1}\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &= \frac{2s}{t}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} z\,dz + 2\sqrt{\frac{(p+1)s}{t}}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz,\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}\int_{p+1}^{\infty} \sqrt{x} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= 2\int_{\sqrt{p+1}}^{\infty} e^{-\frac{\epsilon q'(y-\sqrt{p+1})^2 t}{4s}} y^2 dy \\ &= 2\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} (y+\sqrt{p+1})^2 dy \\ &\le 4\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} y^2 dy + 4(p+1)\int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\le 4\left(\frac{s}{t}\right)^{3/2} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} z^2 dz + 4(p+1)\sqrt{\frac{s}{t}}\int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz\end{aligned} \tag{3.40}$$

Jointly with (3.38), these inequalities imply

$$\sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} \leq C\sqrt{\frac{ps}{t}}. \tag{3.41}$$

Case $N > 2$. Because the value of the right-hand side of (3.36) is an increasing value of N, it is sufficient to prove (3.35) when N is even, say $(N-2)/2 = d \in \mathbb{N}_*$. There holds

$$\sum_{k=-n}^{n} (n+1-|k|)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}} \leq 2\sum_{k=0}^{n} (n+1-k)^d e^{\frac{\epsilon q'(k\sqrt{p+1})t}{2s\sqrt{n}}}. \tag{3.42}$$

We set

$$\alpha = \epsilon q' \frac{t\sqrt{p+1}}{2s\sqrt{n}} \quad \text{and} \quad I_d = \sum_{k=0}^{n} (n+1-k)^d e^{k\alpha}.$$

Since

$$e^{k\alpha} = \frac{e^{(k+1)\alpha} - e^{k\alpha}}{e^\alpha - 1},$$

we use Abel's transform to obtain

$$\begin{aligned} I_d &= \frac{1}{e^\alpha - 1}\left(e^{(n+1)\alpha} - (n+1)^d + \sum_{k=1}^{n} \left((n+2-k)^d - (n+1-k)^d\right) e^{k\alpha} \right) \\ &\leq \frac{1}{e^\alpha - 1}\left((1-d)e^{(n+1)\alpha} - (n+1)^d + de^\alpha \sum_{k=1}^{n} \left((n+1-k)^{d-1}\right) e^{k\alpha} \right). \end{aligned}$$

Therefore the following induction holds

$$I_d \leq \frac{de^\alpha}{e^\alpha - 1} I_{d-1}. \tag{3.43}$$

In (3.37), we have already used the fact that

$$\frac{de^\alpha}{e^\alpha - 1} \leq C\left(1 + \frac{s\sqrt{n}}{t\sqrt{p}}\right),$$

and

$$I_d \leq C\left(1 + \left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right) I_0.$$

Thus (3.38) is replaced by

$$\begin{aligned} \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}} e^{-\frac{\epsilon q' \lambda_{n,j,y}^2}{4s}} &\leq C \sum_{n=p+\ell}^{a_t} \left(1 + \left(\frac{s\sqrt{n}}{t\sqrt{p}}\right)^{d+1}\right) e^{-\frac{\epsilon q'(\sqrt{n}-\sqrt{p+1})^2 t}{4s}} \\ &\leq C\int_{p+1}^{\infty} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx \\ &\qquad + \left(\frac{Cs}{t\sqrt{p}}\right)^{d+1} \int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx. \end{aligned} \tag{3.44}$$

The first integral on the right-hand side has already been estimated in (3.39), for the second integral, there holds

$$\begin{aligned}\int_{p+1}^{\infty} x^{(d+1)/2} e^{-\frac{\epsilon q'(\sqrt{x}-\sqrt{p+1})^2 t}{4s}} dx &= \int_0^{\infty} (y+\sqrt{p+1}\,)^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dx \\ &\leq C\int_0^{\infty} y^{d+2} e^{-\frac{\epsilon q' y^2 t}{4s}} dy + Cp^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' y^2 t}{4s}} dy \\ &\leq C\left(\frac{s}{t}\right)^{2+\frac{d}{2}} \int_0^{\infty} z^{(d+1)/2} e^{-\frac{\epsilon q' z^2}{4}} dz \\ &\qquad + C\left(\frac{s}{t}\right)^{3/2} p^{1+\frac{d}{2}} \int_0^{\infty} e^{-\frac{\epsilon q' z^2}{4}} dz.\end{aligned} \tag{3.45}$$

Combining (3.39), (3.44) and (3.45), we derive (3.35).

Step 2. Since $\mathcal{T}_p^* \subset \Gamma_p \times [0,t]$ where $\Gamma_p = B_{d_{p+1}}(x) \setminus B_{d_{p-1}}(x)$, $(y,s) \in \mathcal{T}_p^*$ implies that $|x-y|^2 \geq (p-1)t$, thus ${J'}_{2,\ell}^{h}$ satisfies

$$\begin{aligned}{J'}_{2,\ell}^{h} &\leq Ct^{\frac{1-q}{2}} \sum_{p=1}^{\infty} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} \\ &\qquad\qquad \times \sum_{n=p+\ell}^{a_t} \sum_{j\in\Theta_{t,n}^h} e^{-\frac{q\lambda_{n,j,y}^2(1-\epsilon)}{4s}} \mu_{n,j}^q(K_{n,j}) ds dy \\ &\leq Ct^{\frac{1-q}{2}} \sum_{n=\ell+1}^{a_t} \sum_{j\in\Theta_{t,n}^h} \mu_{n,j}^q(K_{n,j}) \\ &\quad \times \sum_{p=1}^{n-\ell} p^{\frac{q-1}{2}} \int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-|x-y|^2/4(t-s)} e^{-\frac{q\lambda_{n,j,y}^2(1-\epsilon)}{4s}} ds dy\end{aligned} \tag{3.46}$$

and the constant C depends on N, q and ϵ. Next we set $q_\epsilon = (1-\epsilon)q$. Writting

$$|y-a_{n,j}|^2 = |x-y|^2 + |x-a_{n,j}|^2 - 2\langle y-x, a_{n,j}-x\rangle \geq pt + |x-a_{n,j}|^2 - 2\langle y-x, a_{n,j}-x\rangle,$$

we get

$$\int_{\Gamma_p} e^{-\frac{q_\epsilon |y-a_{n,j}|^2}{4s}} dy = e^{-\frac{q_\epsilon |x-a_{n,j}|^2}{4s}} \int_{\sqrt{tp}}^{\sqrt{t(p+1)}} e^{-\frac{q_\epsilon r^2}{4s}} \int_{|x-y|=r} e^{2q_\epsilon \langle y-x, a_{n,j}-x\rangle/4s} dS_r(y) dr.$$

For estimating the value of the spherical integral, we can assume that $a_{n,j}-x = (0,\ldots,0,|a_{n,j}-x|)$, $y = (y_1,\ldots,y_N)$ and, using spherical coordinates with center at x, that the unit sphere has the representation $S^{N-1} = \{(\sin\phi.\sigma, \cos\phi) \in \mathbb{R}^{N-1}\times\mathbb{R} : \sigma \in S^{N-2},\ \phi \in [0,\pi]\}$. With this representation, $dS_r = r^{N-1}\sin^{N-2}\phi\, d\phi\, d\sigma$ and $\langle y-x, a_{n,j}-x\rangle = |a_{n,j}-x|\,|y-x|\cos\phi$. Therefore

$$\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) = r^{N-1}\left|S^{N-2}\right| \int_0^{\pi} e^{2q_\epsilon \frac{|a_{n,j}-x| r\cos\phi}{4s}} \sin^{N-2}\phi\, d\phi.$$

By Lemma A.3

$$
\begin{aligned}
\int_{|x-y|=r} e^{2q_\epsilon \frac{\langle y-x, a_{n,j}-x\rangle}{4s}} dS_r(y) &\le C \frac{r^{N-1} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}}{\left(1+\frac{r|a_{n,j}-x|}{s}\right)^{\frac{N-1}{2}}} \\
&\le C s^{\frac{N-1}{2}} \left(\frac{r}{|a_{n,j}-x|}\right)^{\frac{N-1}{2}} e^{2q_\epsilon \frac{r|a_{n,j}-x|}{4s}}.
\end{aligned} \tag{3.47}
$$

Therefore

$$
\int_{\Gamma_p} e^{-q_\epsilon \frac{|y-a_{n,j}|^2}{4s}} dy \le C t^{\frac{N-1}{4}} p^{\frac{N-3}{4}} \frac{s^{\frac{N-1}{2}} e^{-q_\epsilon \frac{(|a_{n,j}-x|-\sqrt{t(p+1)})^2}{4s}}}{|a_{n,j}-x|^{\frac{N-1}{2}}}, \tag{3.48}
$$

and, since $|a_{n,j}-x| \ge \sqrt{tn}$,

$$
\begin{aligned}
&\int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-(q(N-1)+1)/2} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon \frac{\lambda_{n,j,y}^2}{4s}} dy\, ds \\
&\quad\le C \frac{\sqrt{t} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^t (t-s)^{-\frac{N}{2}} s^{-\frac{(q-1)(N-1)+1}{2}} e^{-\frac{pt}{4(t-s)}} e^{-q_\epsilon \frac{(\sqrt{tn}-\sqrt{t(p+1)})^2}{4s}} ds \\
&\quad\le C \frac{t^{\frac{1-q(N-1)}{2}} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \int_0^1 (1-s)^{-\frac{N}{2}} s^{-\frac{(q-1)(N-1)+1}{2}} e^{-\frac{p}{4(1-s)}} e^{-q_\epsilon \frac{(\sqrt{n}-\sqrt{p+1})^2}{4s}}.
\end{aligned} \tag{3.49}
$$

We apply Lemma A.1, with $A = \sqrt{p}$, $B = \sqrt{q_\epsilon}(\sqrt{n} - \sqrt{p+1})$, $b = \frac{(q-1)(N-1)+1}{2}$, $a = \frac{N}{2}$ and $\kappa = \sqrt{q_\epsilon}(\ell-1)/8$ as in the case $N=1$, and noticing that, for these specific values,

$$
\begin{aligned}
A^{1-a} B^{1-b} (A+B)^{a+b-2} &= p^{\frac{2-N}{4}} (\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{\frac{1-(q-1)(N-1)}{2}} \\
&\qquad \times (\sqrt{p} + \sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^{\frac{(q-1)(N-1)+N-3}{2}} \\
&\le C \left(\frac{n}{p}\right)^{\frac{N}{4}-1/2} \left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}},
\end{aligned}
$$

where C depends on N, q and κ. Therefore

$$
\begin{aligned}
&\int_0^t \int_{\Gamma_p} (t-s)^{-\frac{N}{2}} s^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4(t-s)}} e^{-q_\epsilon |y-z|^2/4s} dy\, ds \\
&\quad\le C \frac{t^{(1-q(N-1))/2} p^{\frac{N-3}{4}}}{n^{\frac{N-1}{4}}} \left(\frac{n}{p}\right)^{\frac{N}{4}-1/2} \left(\frac{\sqrt{n}-\sqrt{p}}{\sqrt{n}}\right)^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \\
&\quad\le C t^{\frac{1-q(N-1)}{2}} p^{-\frac{1}{4}} n^{\frac{(q-1)(N-1)-2}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned} \tag{3.50}
$$

We derive from (3.46), (3.50),

$$
\begin{aligned}
&{J'}_{2,\ell}^{h} \le C t^{1-\frac{Nq}{2}} \\
&\times \sum_{n=\ell+1}^{a_t} \sum_{j\in\Theta_{t,n}^h} n^{\frac{(q-1)(N-1)-2}{4}} \mu_{n,j}^q(K_{n,j}) \sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n}-\sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}}.
\end{aligned} \tag{3.51}
$$

By Lemma A.2 with $\alpha = \dfrac{2q-3}{4}$, $\beta = \frac{1-(q-1)(N-1)}{2}$, $\delta = \frac{1}{4}$ and $\gamma = q_\epsilon$, we obtain

$$\sum_{p=1}^{n-\ell} p^{\frac{2q-3}{4}} (\sqrt{n} - \sqrt{p})^{\frac{1-(q-1)(N-1)}{2}} e^{-\frac{(\sqrt{p}+\sqrt{q_\epsilon}(\sqrt{n}-\sqrt{p+1}))^2}{4}} \leq C n^{\frac{N(q-1)+q-3}{4}} e^{-\frac{n}{4}}, \tag{3.52}$$

thus

$$J'^{\,h}_{2,\ell} \leq C t^{1-\frac{Nq}{2}} \sum_{n=\ell+1}^{a_t} n^{\frac{N(q-1)}{2}-1} e^{-\frac{n}{4}} \sum_{j\in\Theta^h_{t,n}} \mu^q_{n,j}(K_{n,j}). \tag{3.53}$$

Because

$$\mu_{n,j}(K_{n,j}) = C^{B_{n,j}}_{2/q,q'}(K_{n,j}),$$

we use the rescaling procedure as in the proof of Lemma 2.23, except that the scale factor is $\sqrt{(n+1)t}$ instead of $\sqrt{n+1}$ so that the sets $\tilde{T}_n$, $\tilde{K}_n$, $\tilde{\mathcal{Q}}_n$ and $\tilde{K}_n$ remains unchanged Using again the quasi-additivity and the fact that $J'_{2,\ell} = \sum_{h=1}^{J} J'^{\,h}_{2,\ell}$, we deduce

$$J_{2,\ell} \leq C' t^{-\frac{N}{2}} \sum_{n=\ell+1}^{a_t} d_{n+1}^{N-\frac{2}{q-1}} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{K_n}{d_{n+1}}\right), \tag{3.54}$$

which implies (3.32). □

The proof of Theorem 3.1 follows from the previous estimates on J_1 and J_2. Furthermore the following integral expression holds

Theorem 3.8 *Assume $q \geq q_c$. Then there exists a positive constants C_2^* , depending on N,q and T, such that for any closed set F, there holds*

$$\underline{u}_F(x,t) \geq \frac{C_2^*}{t^{1+\frac{N}{2}}} \int_0^{\sqrt{t a_t}} e^{-\frac{s^2}{4t}} s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds, \tag{3.55}$$

where a_t is the smallest integer j such that $F \subset B_{\sqrt{jt}}(x)$.

Proof. We distinguish according $q = q_c$, or $q > q_c$, and for simplicity we denote $B_r = B_r(x)$ for the various values of r.
Case 1: $q = q_c \iff N - \frac{2}{q-1} = 0$. Because $F_n = F \cap (B_{d_{n+1}} \setminus B_{d_n})$ there holds

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - C_{2/q,q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right),$$

Furthermore, since $d_{n+1} \geq d_n$,

$$C_{2/q,q'}\left(\frac{F \cap B_{d_n}}{d_{n+1}}\right) = C_{2/q,q'}\left(\frac{d_n}{d_{n+1}} \frac{F \cap B_{d_n}}{d_n}\right) \leq C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right),$$

thus

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}} \cap B_1\right) - C_{2/q,q'}\left(\frac{F}{d_n} \cap B_1\right),$$

it follows

$$\begin{aligned}\sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right) - \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_n}\cap B_1\right)\\ &\geq \sum_{n=1}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right) - e^{-\frac{1}{4}}\sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right)\\ &\geq (1-e^{-\frac{1}{4}})\sum_{n=1}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right) - e^{-\frac{1}{4}} C_{2/q,q'}\left(\frac{F}{\sqrt{t}}\cap B_1\right).\end{aligned}$$

Since, by (2.92),

$$C_{2/q,q'}\left(\frac{F}{s'}\cap B_1\right) \geq C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right) \geq C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right),$$

for any $s' \in [d_{n+1}, d_{n+2}]$ and $s \in [d_n, d_{n+1}]$, there holds

$$\begin{aligned}te^{-\frac{n}{4}} C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right) &\geq C_{2/q,q'}\left(\frac{F}{d_{n+1}}\cap B_1\right)\int_{d_n}^{d_{n+1}} e^{-s^2/4t} s\,ds\\ &\geq \int_{d_n}^{d_{n+1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right) s\,ds.\end{aligned}$$

This implies

$$W_F(x,t) \geq (1-e^{-\frac{1}{4}})t^{-(1+\frac{N}{2})}\int_0^{\sqrt{ta_t}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s}\cap B_1\right) s\,ds.$$

Case 2: $q > q_c \Longleftrightarrow N - \frac{2}{q-1} > 0$. In that case it follows from Lemma 2.9 that

$$C_{2/q,q'}\left(\frac{F_n}{d_{n+1}}\right) \approx d_{n+1}^{\frac{2}{q-1}-N} C_{2/q,q'}(F_n).$$

Thus

$$W_F(x,t) \approx t^{-1-\frac{N}{2}}\sum_{n=0}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}(F_n).$$

Since

$$C_{2/q,q'}(F_n) \geq C_{2/q,q'}\left(F\cap B_{d_{n+1}}\right) - C_{2/q,q'}(F\cap B_{d_n}),$$

and again

$$\begin{aligned}t^{-\frac{N}{2}}\sum_{n=0}^{a_t} e^{-\frac{n}{4}} C_{2/q,q'}(F_n) &\geq (1-e^{-\frac{1}{4}})t^{-\frac{N}{2}}\sum_{n=0}^{a_t-1} e^{-\frac{n}{4}} C_{2/q,q'}\left(F\cap B_{d_{n+1}}\right)\\ &\geq (1-e^{-\frac{1}{4}})t^{-(1+\frac{N}{2})}\int_0^{\sqrt{ta_t}} e^{-\frac{s^2}{4t}} C_{2/q,q'}(F\cap B_s)\,s\,ds.\end{aligned}$$

Because $C_{2/q,q'}(F\cap B_s) \approx s^{N-\frac{2}{q-1}} C_{2/q,q'}\left(s^{-1}F\cap B_1\right)$, (3.55) follows. □

4 Applications

The first result of this section is the following

Theorem 4.1 *Assume $N \geq 1$ and $q > 1$. Then $\overline{u}_K = \underline{u}_K$.*

Proof. If $1 < q < q_c$, the result is already proved in [28]. The proof in the super-critical case is an adaptation that we recall, for the sake of completeness. By Theorem 2.24 and Theorem 3.8 there exists a positive constant C, depending on N, q and T such that

$$\overline{u}_F(x,t) \leq C\underline{u}_F(x,t) \qquad \forall (x,t) \in Q_T.$$

By convexity $\tilde{u} = \underline{u}_F - \dfrac{1}{2C}(\overline{u}_F - \underline{u}_F)$ is a super-solution, which is smaller than $\underline{u}_F$ if we assume that $\overline{u}_F \neq \underline{u}_F$. If we set $\theta := 1/2 + 1/(2C)$, then $u_\theta = \theta \overline{u}_F$ is a subsolution. Therefore there exists a solution u_1 of (1.1) in Q_∞ such that $u_\theta \leq u_1 \leq \tilde{u} < \underline{u}_F$. If $\mu \in \mathfrak{M}^q_+(\mathbb{R}^N)$ satisfies $\mu(F^c) = 0$, then $u_{\theta\mu}$ is the smallest solution of (1.1) which is above the subsolution θu_μ. Thus $u_{\theta\mu} \leq u_1 < \underline{u}_F$ and finally $\underline{u}_F \leq u_1 < \underline{u}_F$, a contradiction. $\square$

If we combine Theorem 2.24 and Theorem 3.8 we derive the following integral approximation of the parabolic capacitary potential

Proposition 4.2 *Assume $q \geq q_c$. Then there exist two positive constants $C_1^\dagger$, $C_2^\dagger$, depending only on N, q and T such that*

$$\begin{aligned} C_2^\dagger t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \leq W_F(x,t) \\ \leq C_1^\dagger t^{-(1+\frac{N}{2})} \int_{\sqrt{t}}^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \end{aligned} \tag{4.56}$$

for any $(x,t) \in Q_T$.

Definition 4.3 *If F is a closed subset of $\mathbb{R}^N$, we define the $(2/q,q')$-integral parabolic capacitary potential $\mathcal{W}_F$ by*

$$\mathcal{W}_F(x,t) = t^{-1-\frac{N}{2}} \int_0^{D_F(x)} s^{N-\frac{2}{q-1}} e^{-s^2/4t} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \qquad \forall (x,t) \in Q_\infty, \tag{4.57}$$

where $D_F(x) = \max\{|x-y| : y \in F\}$.

An easy computation shows that

$$\begin{aligned} 0 \leq \mathcal{W}_F(x,t) - t^{-(1+\frac{N}{2})} \int_0^{\sqrt{ta_t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds \\ \leq C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-D_F^2(x)/4t}, \end{aligned} \tag{4.58}$$

and

$$0 \le t^{-(1+\frac{N}{2})} \int_0^{\sqrt{t(a_t+2)}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4t}} C_{2/q,q'}\left(\frac{F}{s} \cap B_1(x)\right) s\,ds - \mathcal{W}_F(x,t) \le C \frac{t^{(q-3)/2(q-1)}}{D_F(x)} e^{-\frac{D_F^2(x)}{4t}}, \tag{4.59}$$

for some $C = C(N,q) > 0$. Furthermore

$$\mathcal{W}_F(x,t) = t^{-\frac{1}{q-1}} \int_0^{D_F(x)/\sqrt{t}} s^{N-\frac{2}{q-1}} e^{-\frac{s^2}{4}} C_{2/q,q'}\left(\frac{F}{s\sqrt{t}} \cap B_1(x)\right) s\,ds. \tag{4.60}$$

The following result gives a sufficient condition in order that $\overline{u}_F$ does not have a strong blow-up at a point x.

Proposition 4.4 *Assume $q \ge q_c$ and F is a closed subset of $\mathbb{R}^N$. If there exists $\gamma \in [0,\infty)$ such that*

$$\lim_{\tau \to 0} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) = \gamma, \tag{4.61}$$

then

$$\lim_{t \to 0} t^{\frac{1}{q-1}} \overline{u}_F(x,t) = C\gamma, \tag{4.62}$$

for some $C = C(N,q) > 0$.

Proof. Clearly, condition (4.61) implies

$$\lim_{t \to 0} C_{2/q,q'}\left(\frac{F}{\sqrt{t}s} \cap B_1(x)\right) = \gamma$$

for any $s > 0$. Then (4.62) follows by Lebesgue's theorem. Notice also that the set of γ is bounded from above by a constant depending on N and q. □

In the next result we give a condition in order that the solution remains bounded at a point x. The proof is similar to the previous one.

Proposition 4.5 *Assume $q \ge q_c$ and F is a closed subset of $\mathbb{R}^N$. If*

$$\limsup_{\tau \to 0} \tau^{-\frac{2}{q-1}} C_{2/q,q'}\left(\frac{F}{\tau} \cap B_1(x)\right) < \infty, \tag{4.63}$$

then $\overline{u}_F(x,t)$ remains bounded when $t \to 0$.

Remark. If we assume that f is a convex function on $\mathbb{R}^+$ satisfying

$$c_2 r^q \le f(r) \le c_1 r^q \qquad \forall r \ge 0 \tag{4.64}$$

for some $0 < c_2 \le c_1$ we can construct in the same way as for (1.1) the solutions $\underline{u}_F$ and $\overline{u}_F$ for equation

$$\partial_t u - \Delta u + f(u) = 0 \qquad \text{in } Q_T. \tag{4.65}$$

The bilateral estimate estimate (1.19) is still valid (up to change of the C_i). Since only convexity of f is used in the proof of Theorem 4.1, there still holds $\underline{u}_F = \overline{u}_F$. Similar extensions of Proposition 4.4 and Proposition 4.5 are also clear.

A Appendix

The next estimate is crucial in our study of semilinear parabolic equations.

Lemma A.1 *Let a and b be two real numbers, $a > 0$ and $\kappa > 0$. Then there exists a constant $C = C(a, b, \kappa) > 0$ such that for any $A > 0$, $B > \kappa/A$ there holds*

$$\int_0^1 (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x} dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \tag{A.1}$$

Proof. We first notice that

$$\max\{e^{-A^2/4(1-x)} e^{-B^2/4x} : 0 \leq x \leq 1\} = e^{-(A+B)^2/4}, \tag{A.2}$$

and it is achieved for $x_0 = B/(A+B)$. Set $\Phi(x) = (1-x)^{-a} x^{-b} e^{-A^2/4(1-x)} e^{-B^2/4x}$, thus

$$\int_0^1 \Phi(x) dx = \int_0^{x_0} \Phi(x) dx + \int_{x_0}^1 \Phi(x) dx = I_{a,b} + J_{a,b}.$$

Put

$$u = \frac{A^2}{4(1-x)} + \frac{B^2}{4x}, \tag{A.3}$$

then

$$4ux^2 - (4u + B^2 - A^2)x + B^2 = 0. \tag{A.4}$$

If $0 < x < x_0$ this equation admits the solution

$$x = x(u) = \frac{1}{8u}\left(4u + B^2 - A^2 - \sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}\right)$$

$$\int_0^{x_0} (1-x)^{-a} x^{-b} e^{-A^2/4(1-x) - B^2/4x} dx = -\int_{(A+B)^2/4}^{\infty} (1-x(u))^{-a} x(u)^{-b} e^{-u} x'(u) du$$

Putting $x' = x'(u)$ and differentiating $(A.4)$,

$$4x^2 + 8uxx' - (4u + B^2 - A^2)x' - 4x = 0 \Longrightarrow -x' = \frac{4x(1-x)}{4u + B^2 - A^2 - 8ux}.$$

Thus

$$\int_0^{x_0} \Phi(x) dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1} x(u)^{-b+1} e^{-u} du}{4u + B^2 - A^2 - 8ux(u)}. \tag{A.5}$$

Using the explicit value of the root $x(u)$, we finally get

$$\int_0^{x_0} \Phi(x) dx = 4\int_{(A+B)^2/4}^{\infty} \frac{(1-x(u))^{-a+1} x(u)^{-b+1} e^{-u} du}{\sqrt{16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2}}, \tag{A.6}$$

and the factorization below holds

$$16u^2 - 8u(A^2+B^2) + (A^2-B^2)^2 = 16(u - (A+B)^2/4)(u - (A-B)^2/4).$$

We set $u = v + (A+B)^2/4$ and obtain

$$x(u) = \frac{v + (AB + B^2)/2 - \sqrt{v(v+AB)}}{2\,(v + (A+B)^2/4)},$$

and

$$1 - x(u) = \frac{v + (A^2 + AB)/2 + \sqrt{v(v+AB)}}{2\,(v + (A+B)^2/4)}.$$

We introduce the relation $\approx$ linking two positive quantities depending on A and B. It means that the two sided-inequalities up to multiplicative constants independent of A and B. Therefore

$$\int_0^{x_0} \Phi(x)dx = 2^{a-b-4} e^{-(A+B)^2/4} \int_0^\infty \tilde{\Phi}(v)dv \quad \text{where}$$

$$\tilde{\Phi}(v) = \frac{\left(v + (AB+B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b} \left(v + (A^2+AB)/2 + \sqrt{v(v+AB)}\right)^{1-a}}{(v + (A+B)^2/4)^{2-a-b} \sqrt{v(v+AB)}} e^{-v}dv. \tag{A.7}$$

Case 1: $a \geq 1$, $b \geq 1$. First

$$\frac{\left(v + (A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+AB)}} \leq \frac{\left(v + (A+B)^2/4\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \approx \frac{\left(v + (A+B)^2\right)^{a+b-2}}{\sqrt{v(v+\kappa)}} \tag{A.8}$$

since $a + b - 2 \geq 0$ and $AB \geq \kappa$. Next

$$\left(v + (A^2+AB)/2 + \sqrt{v(v+AB)}\right)^{1-a} \approx (v + A(A+B))^{1-a}. \tag{A.9}$$

Furthermore

$$\begin{aligned} v + (AB+B^2)/2 - \sqrt{v(v+AB)} &= B^2 \frac{v + (A+B)^2/4}{v + B(A+B)/2 + \sqrt{v(v+AB)}} \\ &\approx B^2 \frac{v + (A+B)^2}{v + B(A+B)}. \end{aligned} \tag{A.10}$$

Then

$$\left(v + (AB+B^2)/2 - \sqrt{v(v+AB)}\right)^{1-b} \approx B^{2-2b} \left(\frac{v + B(A+B)}{v + (A+B)^2}\right)^{b-1} \tag{A.11}$$

It follows

$$\begin{aligned} \tilde{\Phi}(v) &\leq CB^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)}\right)^{a-1} \frac{(v + B(A+B))^{b-1}}{\sqrt{v(v+\kappa)}} \\ &\leq CB^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)}\right)^{a-1} \frac{v^{b-1} + (B^2 + AB)^{b-1}}{\sqrt{v(v+\kappa)}} \end{aligned} \tag{A.12}$$

where C depends on a, b and κ. The function $v \mapsto (v + (A+B)^2)/(v + A(A+B))$ is decreasing on $(0, \infty)$. If we set

$$C_1 = \int_0^\infty \frac{v^{b-1} e^{-v} dv}{\sqrt{v(v+\kappa)}} \quad \text{and} \quad C_2 = \int_0^\infty \frac{e^{-v} dv}{\sqrt{v(v+\kappa)}}$$

then

$$C_1 \leq K(B^2 + AB)^{b-1} C_2$$

with $K = C_1 \kappa^{1-b}/C_2$. Therefore

$$\int_0^{x_0} \Phi(x)dx \leq C e^{-(A+B)^2/4} B^{1-b} A^{1-a} (A+B)^{a+b-2}. \tag{A.13}$$

The estimate of $J_{a,b}$ is obtained by exchanging (A,a) with (B,b) and replacing x by $1-x$. *Mutadis mutandis*, this yields directely to the same expression as in A.13 and finally

$$\int_0^{1} \Phi(x)dx \leq C e^{-(A+B)^2/4} A^{1-a} B^{1-b} (A+B)^{a+b-2}. \tag{A.14}$$

Case 2: $a \geq 1$, $b < 1$. Estimates $(A.7)$, $(A.8)$, $(A.9)$, $(A.10)$ and $(A.11)$ are valid. Because $v \mapsto (v + B(A+B))^{b-1}$ is decreasing, $(A.12)$ has to be replaced by

$$\tilde{\Phi}(v) \leq C B^{2-2b} \left(\frac{v + (A+B)^2}{v + A(A+B)}\right)^{a-1} \frac{\left(AB + B^2\right)^{b-1}}{\sqrt{v(v+\kappa)}}. \tag{A.15}$$

This implies $(A.13)$ directly. The estimate of $J_{a,b}$ is performed by the change of variable $x \mapsto 1-x$. If $x_1 = 1 - x_0$, there holds

$$J_{a,b} = \int_0^{x_1} x^{-a}(1-x)^{-b} e^{-A^2/4x} e^{-B^2/4(1-x)} dx = \int_0^{x_1} \Psi(x)dx.$$

Then

$$\int_0^{x_1} \Psi(x)dx = 2^{b-a-4} e^{-(A+B)^2/4} \int_0^{x_1} \tilde{\Psi}(v)dv \quad \text{where}$$

$$\tilde{\Psi}(v) = \frac{\left(v + (AB + A^2)/2 - \sqrt{v(v+AB)}\right)^{1-a} \left(v + (B^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-b}}{\left(v + (A+B)^2/4\right)^{2-a-b} \sqrt{v(v+AB)}} e^{-v} dv. \tag{A.16}$$

Equivalence $(A.8)$ is unchanged; $(A.9)$ is replaced by

$$\left(v + (B^2 + AB)/2 + \sqrt{v(v+AB)}\right)^{1-b} \approx (v + B(A+B))^{1-b}, \tag{A.17}$$

$(A.10)$ by

$$v + (AB + A^2)/2 - \sqrt{v(v+AB)} \approx A^2 \frac{v + (A+B)^2}{v + A(A+B)}, \tag{A.18}$$

and $(A.11)$ by

$$\left(v + (AB + A^2)/2 - \sqrt{v(v+AB)}\right)^{1-a} \approx A^{2-2a} \left(\frac{v + A(A+B)}{v + (A+B)^2}\right)^{a-1}. \tag{A.19}$$

Because $a > 1$, $(A.12)$ turns into

$$\begin{aligned}\tilde{\Psi}(v) &\leq CA^{2-2b}(v+(A+B)^2)^{b-1}\frac{(v+A^2+AB)^{a-1}(v+B^2+AB)^{1-b}}{\sqrt{v(v+\kappa)}}\\ &\leq Ce^{-(A+B)^2/4}A^{2-2b}(A+B)^{2b-2}\\ &\qquad\times\frac{v^{a-b}+(A^2+AB)^{a-1}v^{1-b}+(B^2+AB)^{1-b}v^{a-1}+A^{a-1}B^{1-b}(A+B)^{a-b}}{\sqrt{v(v+\kappa)}}.\end{aligned} \tag{A.20}$$

Because $AB \geq \kappa$, there exists a positive constant C, depending on κ, such that

$$\begin{aligned}\int_0^\infty &\frac{v^{a-b}+(A^2+AB)^{a-1}v^{1-b}+(B^2+AB)^{1-b}v^{a-1}}{\sqrt{v(v+\kappa)}}e^{-v}dv\\ &\leq CA^{a-1}B^{1-b}(A+B)^{a-b}\int_0^\infty\frac{e^{-v}dv}{\sqrt{v(v+\kappa)}}.\end{aligned} \tag{A.21}$$

Combining $(A.20)$ and $(A.21)$ yields to

$$\int_0^{x_1}\Psi(x)dx \leq Ce^{-(A+B)^2/4}A^{1-a}B^{1-b}(A+B)^{a+b-2}. \tag{A.22}$$

This, again, implies that $(A.1)$ holds.

Case 3: $\max\{a,b\} < 1$. Inequalities $(A.7)$-$(A.11)$ hold, but $(A.12)$ has to be replaced by

$$\begin{aligned}\tilde{\Phi}(v) &\leq CB^{2-2b}\left(\frac{v+(A+B)^2}{v+A(A+B)}\right)^{a-1}\frac{\left(v+B^2+AB\right)^{b-1}}{\sqrt{v(v+\kappa)}}\\ &\leq CB^{1-b}(A+B)^{2a+b-3}\frac{v^{1-a}+\left(A^2+AB\right)^{1-a}}{\sqrt{v(v+\kappa)}}\end{aligned} \tag{A.23}$$

Noticing that

$$\int_0^\infty\frac{v^{1-a}e^{-v}dv}{\sqrt{v(v+\kappa)}} \leq C\left(A^2+AB\right)^{1-a}\int_0^\infty\frac{e^{-v}dv}{\sqrt{v(v+\kappa)}},$$

it follows that $(A.13)$ holds. Finally $(A.14)$ holds by exchanging (A,a) and (B,b). □

Lemma A.2 . *Let α, β, γ, δ be real numbers and ℓ an integer. We assume $\gamma > 1$, $\delta > 0$ and $\ell \geq 2$. Then there exists a positive constant C such that, for any integer $n > \ell$*

$$\sum_{p=1}^{n-\ell}p^\alpha(\sqrt{n}-\sqrt{p})^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \leq Cn^{\alpha-\beta/2}e^{-\delta n}. \tag{A.24}$$

Proof. The function $x \mapsto (\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2$ is decreasing on $[(\gamma-1)^{-1},\infty)$. Furthermore there exists $C > 0$ depending on ℓ, α and β such that $p^\alpha(\sqrt{n}-\sqrt{p})^\beta \leq Cx^\alpha(\sqrt{n}-\sqrt{x+1})^\beta$

for $x \in [p, p+1]$ If we denote by p_0 the smallest integer larger than $(\gamma-1)^{-1}$, we derive

$$
\begin{aligned}
S = \sum_{p=1}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2/4} &= \sum_{p=1}^{p_0-1} + \sum_{p_0}^{n-\ell} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \\
\leq \sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} & \\
&+ C \int_{p_0}^{n+1-\ell} x^\alpha (\sqrt{n} - \sqrt{x}\,)^\beta e^{-\delta(\sqrt{x}+\sqrt{\gamma}(\sqrt{n}-\sqrt{x+1}))^2} dx,
\end{aligned}
$$

(notice that $\sqrt{n} - \sqrt{x} \approx \sqrt{n} - \sqrt{x+1}$ for $x \leq n - \ell$). Clearly

$$
\sum_{p=1}^{p_0-1} p^\alpha (\sqrt{n} - \sqrt{p}\,)^\beta e^{-\delta(\sqrt{p}+\sqrt{\gamma}(\sqrt{n}-\sqrt{p+1}))^2} \leq C_0 n^\alpha (\sqrt{n} - \sqrt{n-\ell}\,)^\beta e^{-\delta n} \tag{A.25}
$$

for some C_0 independent of n. We set $y = y(x) = \sqrt{x+1} - \sqrt{x}/\sqrt{\gamma}$. Obviously

$$
y'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{\gamma}\sqrt{x}} \right) \qquad \forall x \geq p_0,
$$

and their exists $\epsilon = \epsilon(\delta, \gamma) > 0$ such that $\sqrt{2}\sqrt{x} \geq y(x) \geq \epsilon\sqrt{x}$ and $y'(x) \geq \epsilon/\sqrt{x}$. Furthermore

$$
\sqrt{x} = \frac{\sqrt{\gamma}\left(y + \sqrt{\gamma y^2 + 1 - \gamma} \right)}{\gamma - 1},
$$

$$
\begin{aligned}
\sqrt{n} - \sqrt{x} &= \frac{\sqrt{n}(\gamma-1) - \sqrt{\gamma} y - \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}}{\gamma - 1} \\
&= \frac{n(\gamma-1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}(\gamma-1) - \sqrt{\gamma} y + \sqrt{\gamma}\sqrt{\gamma y^2 + 1 - \gamma}} \\
&\approx \frac{n(\gamma-1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2}{\sqrt{n}}
\end{aligned}
$$

since $y(x) \leq \sqrt{n}$. Furthermore

$$
\begin{aligned}
n(\gamma-1) + \gamma - 2y\sqrt{\gamma n} - \gamma y^2 &= \gamma(\sqrt{n+1} + \sqrt{n}/\sqrt{\gamma} + y)(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y) \\
&\approx \sqrt{n}(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y),
\end{aligned}
$$

because y ranges between $\sqrt{n+2-\ell} - \sqrt{n+1-\ell}\sqrt{\gamma} \approx \sqrt{n}$ and $\sqrt{p_0+1} - \sqrt{p_0}\sqrt{\gamma}$. Thus

$$
(\sqrt{n} - \sqrt{x}\,)^\beta \approx \left(\sqrt{n+1} - \sqrt{n}/\sqrt{\gamma} - y \right)^\beta .
$$

This implies

$$\int_{p_0}^{n+1-\ell} x^\alpha(\sqrt{n}-\sqrt{x}\,)^\beta e^{-\delta(\sqrt{x}+\gamma(\sqrt{n}-\sqrt{x+1}))^2}dx$$

$$\le C\int_{y(p_0)}^{y(n+1-\ell)} y^{2\alpha+1}\left(\sqrt{n+1}-\sqrt{n}/\sqrt{\gamma}-y\right)^\beta e^{-\gamma\delta(\sqrt{n}-y)^2}dy$$

$$\le Cn^{\alpha+\beta/2+1}\int_{1-y(n+1-\ell)/\sqrt{n}}^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz. \tag{A.26}$$

Moreover

$$1-\frac{y(p_0)}{\sqrt{n}}=1-\frac{1}{\sqrt{n}}\left(\sqrt{p_0+1}-\frac{\sqrt{p_0}}{\sqrt{\gamma}}\right),$$

$$\begin{aligned}1-\frac{y(n-\ell+1)}{\sqrt{n}}&=1-\frac{\sqrt{n-\ell+2}}{\sqrt{n}}+\frac{\sqrt{n-\ell+1}}{\sqrt{n\gamma}}\\&=\frac{1}{\sqrt{\gamma}}\left(1+\frac{\sqrt{\gamma}\,(\ell-2)-\ell+1}{2n}+\frac{\sqrt{\gamma}\,(\ell-2)^2-(\ell-1)^2}{8n^2}\right)+O(n^{-3}).\end{aligned} \tag{A.27}$$

Let θ fixed such that $1-\dfrac{y(n-\ell+1)}{\sqrt{n}}<\theta<1-\dfrac{y(p_0)}{\sqrt{n}}$ for any $n>p_0$. Then

$$\begin{aligned}\int_\theta^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz&\le C_\theta\int_\theta^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}e^{-\gamma\delta n z^2}dz\\&\le C_\theta\, e^{-\gamma\delta n\theta^2}\int_\theta^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}dz\\&\le C\, e^{-\gamma\delta n\theta^2}\max\{1,n^{-\alpha-1/2}\}.\end{aligned}$$

Because $\gamma\theta^2>1$ we derive

$$\int_\theta^{1-y(p_0)/\sqrt{n}}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz\le Cn^{-\beta}e^{-\delta n}, \tag{A.28}$$

for some constant $C>0$. On the other hand

$$\begin{aligned}&\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}(1-z)^{2\alpha+1}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz\\&\qquad\le C'_\theta\int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta}(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta e^{-\gamma\delta n z^2}dz.\end{aligned}$$

The minimum of $z\mapsto(z+\sqrt{1+1/n}-1-1/\sqrt{\gamma})^\beta$ is achieved at $1-y(n+1-\ell)$ with value

$$\frac{\sqrt{\gamma}(\ell+1)+1-\ell}{2n\sqrt{\gamma}}+O(n^{-2}),$$

and the maximum of the exponential term is achieved at the same point with value

$$e^{-n\delta+((\ell-2)\sqrt{\gamma}+1-\ell)/2}(1+\circ(1))=C_\gamma e^{-n\delta}(1+\circ(1)).$$

We denote

$$z_{\gamma,n} = 1 + 1/\sqrt{\gamma} - \sqrt{1+1/n} \quad \text{and} \quad I_\beta = \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^\beta e^{-\gamma\delta n z^2} dz.$$

Since $1 - y(n+1-\ell) \geq 1/\sqrt{2\gamma}$ for n large enough,

$$\begin{aligned} I_\beta &\leq \sqrt{2\gamma} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^\beta z e^{-\gamma\delta n z^2} dz \\ &\leq \frac{-\sqrt{2\gamma}}{2n\gamma\delta} \left[(z - z_{\gamma,n})^\beta e^{-\gamma\delta n z^2} \right]_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} + \frac{\beta\sqrt{2\gamma}}{2n\gamma\delta} \int_{1-y(n+1-\ell)/\sqrt{n}}^{\theta} (z - z_{\gamma,n})^{\beta-1} z e^{-\gamma\delta n z^2} dz \end{aligned}$$

But $1 - y(n+1-\ell)/\sqrt{n} - z_{\gamma,n} = (\ell - 1)(1 - 1/\sqrt{\gamma})/2n$, therefore

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + \beta C_1' n^{-1} I_{\beta-1}. \tag{A.29}$$

If $\beta \leq 0$, we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n},$$

which inequality, combined with $(A.26)$ and $(A.28)$, yields to $(A.24)$. If $\beta > 0$, we iterate and get

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C_1' n^{-1} (C_1 n^{-\beta} e^{-\delta n} + (\beta - 1) C_1' n^{-1} I_{\beta-2})$$

If $\beta - 1 \leq 0$ we derive

$$I_\beta \leq C_1 n^{-\beta-1} e^{-\delta n} + C_1 C_1' n^{-1-\beta} e^{-\delta n} = C_2 n^{-\beta-1} e^{-\delta n},$$

which again yields to $(A.24)$. If $\beta - 1 > 0$, we continue up we find a positive integer k such that $\beta - k \leq 0$, which again yields to

$$I_\beta \leq C_k n^{-\beta-1} e^{-\delta n}$$

and to $(A.24)$. □

The next estimate is fundamental in deriving the N-dimensional estimate.

Lemma A.3 *For any integer $N \geq 2$ there exists a constant $c_N > 0$ such that*

$$\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta \leq c_N \frac{e^m}{(1+m)^{(N-1)/2}} \qquad \forall m > 0. \tag{A.30}$$

Proof. Put $\mathcal{I}_N(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \sin^{N-2}\theta \, d\theta$. Then $\mathcal{I}_2'(m) = \displaystyle\int_0^\pi e^{m\cos\theta} \cos\theta \, d\theta$ and

$$\begin{aligned} \mathcal{I}_2''(m) &= \int_0^\pi e^{m\cos\theta} \cos^2\theta \, d\theta = \mathcal{I}_2(m) - \int_0^\pi e^{m\cos\theta} \sin^2\theta \, d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m} \int_0^\pi e^{m\cos\theta} \cos\theta \, d\theta \\ &= \mathcal{I}_2(m) - \frac{1}{m} \mathcal{I}_2'(m). \end{aligned}$$

Thus $\mathcal{I}_2$ satisfies a Bessel equation of order 0. Since $\mathcal{I}_2(0) = \pi$ and $\mathcal{I}_2'(0) = 0$, $\pi^{-1}\mathcal{I}_2$ is the modified Bessel function of index 0 (usually denoted by I_0) the asymptotic behaviour of which is well known, thus $(A.30)$ holds. If $N = 3$

$$\mathcal{I}_3(m) = \int_0^\pi e^{m\cos\theta} \sin\theta \, d\theta = \left[\frac{-e^{m\cos\theta}}{m}\right]_0^\pi = \frac{2\sinh m}{m}.$$

For $N > 3$ arbitrary

$$\mathcal{I}_N(m) = \int_0^\pi \frac{-1}{m}\frac{d}{d\theta}(e^{m\cos\theta})\sin^{N-3}\theta \, d\theta = \frac{N-3}{m}\int_0^\pi e^{m\cos\theta}\cos\theta\sin^{N-4}\theta \, d\theta. \tag{A.31}$$

Therefore,

$$\mathcal{I}_4(m) = \frac{1}{m}\int_0^\pi e^{m\cos\theta}\cos\theta \, d\theta = \mathcal{I}_2'(m),$$

and, again $(A.30)$ holds since $I_0'(m)$ has the same behaviour as $I_0(m)$ at infinity. For $N \geq 5$

$$\mathcal{I}_N(m) = \frac{3-N}{m^2}\left[e^{m\cos\theta}\cos\theta\sin^{N-5}\theta\right]_0^\pi + \frac{N-3}{m^2}\int_0^\pi e^{m\cos\theta}\frac{d}{d\theta}\left(\cos\theta\sin^{N-5}\theta\right)d\theta.$$

Differentiating $\cos\theta\sin^{N-5}\theta$ and using $(A.31)$, we obtain

$$\mathcal{I}_5(m) = \frac{4\sinh m}{m^2} - \frac{4\sinh m}{m^3},$$

while

$$\mathcal{I}_N(m) = \frac{(N-3)(N-5)}{m^2}\left(\mathcal{I}_{N-4}(m) - \mathcal{I}_{N-2}(m)\right), \tag{A.32}$$

for $N \geq 6$. Since the estimate $(A.30)$ for $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$ and $\mathcal{I}_5$ has already been obtained, a straigth-forward induction yields to the general result. □

Remark. Although it does not has any importance for our use, it must be noticed that $\mathcal{I}_N$ can be expressed either with hyperbolic functions if N is odd, or with Bessel functions if N is even.

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